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ABSTRACT

This book is part 2 of a 2-part manual for teachers using SMSG text materials for grade 6. The purpose for each of 5 chapters is stated and mathematical background for the teacher is presented. Detailed lesson plans are then provided, including sequences of statements and questions, activities, and exercise sets with answers. Needed materials and vocabulary are listed. Chapter topics include: (1) division of rational numbers; (2) volume; (3) organizing and describing data; (4) sets and circles; and (5) a review. (MN)

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TEACHER'S COMMENTARY UNIT NO. 36

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MATHEMATICS FOR THE ELEMENTARY SCHOOL
GRADE 6
PART II

SE 025 384

SCHOOL MATHEMATICS STUDY GROUP

Mathematics for the Elementary School, Grade 6

Teacher's Commentary, Part II

REVISED EDITION

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Chapter 6

DIVISION OF RATIONAL NUMBERS

PURPOSE OF THE UNIT

The purposes of the unit are these:

1) To develop understanding of the operation of division as applied to rational numbers.

2) To develop methods for computing quotients of rational numbers using fraction numerals and decimal numerals.

3) To extend the meaning of the fraction symbol to include

    a) the use of fractions whose numerators and denominators are rational numbers.

    b) The use of the fraction symbol to indicate the quotient of two whole numbers or of two rational numbers.

4) To develop understanding of the limitations of a place-value system of number notation for naming rational numbers.

5) To provide experience in solving problems requiring division of rational numbers.

OVERVIEW OF THE UNIT

The sections in this unit may be grouped in five parts, as follows:

I. Review

- Some Facts You Know About Rational Numbers
- Common Denominators
- Decimal Names for Rational Numbers
- Properties of Rational Numbers

II. The Operation of Division of Rational Numbers and Computing Quotients, Using Fraction Numerals

- Getting Ready for Division of Rational Numbers
- Division by a Rational Number
- Computing Quotients of Rational Numbers

III. Application of Division of Rational Numbers to Problem-Solving

- Problems Solved by Division
- Division on the Number Line

IV. New Meanings for the Fraction Symbol

- Rational Numbers as Quotients of Whole Numbers
- Extending the System of Fractions

V. Using Decimal Numerals in Division

- Division of Rational Numbers Named by Decimals
- Extending the Division Process
- Estimating Rational Numbers Using Decimals

It is of course desirable that the teacher be familiar with the entire unit before beginning to teach it. It is essential that he be thoroughly familiar with all of the sections under one of the five main topics before beginning the first section on that topic.

In this Commentary, comments on all of the sections pertaining to one of the main topics are combined, and precede the related pages of the pupil text.

Teaching the Unit

I. SOME FACTS YOU KNOW ABOUT RATIONAL NUMBERS

COMMON DENOMINATORS

DECIMAL NAMES FOR RATIONAL NUMBERS

PROPERTIES OF RATIONAL NUMBERS

PICTURING RATIONAL NUMBERS

Objective: Since this is the last chapter on rational numbers for the year, it is the purpose of this section to summarize and reteach, when necessary, the following concepts of rational numbers developed in previous chapters in Grades Four, Five, and Six:

(1) Rational numbers are used as measures of part of a unit segment, region, or set.

(2) A rational number has many names: fractions, mixed forms, and decimals.

(3) Rational numbers have many of the properties of whole numbers. The reciprocal property is a special property of every rational number except zero.

(4) The operations of addition, subtraction, and multiplication can be performed on rational numbers.

(5) Many problems require the use of rational numbers.

Materials: Number lines, models of circular and square regions, arrays, pocket chart and cards for place value of decimals, as needed.

Suggested Teaching Procedure:

See Mathematical Background for Grade 6, Chapter 2.

The teacher should be thoroughly familiar with the texts for Grades 4 and 5, especially Grade 4, Chapter 10, and Grade 5, Chapters 1, 2, and 6.

As an informal inventory test of understanding of rational numbers, have children suggest words to build a "Mathematical Vocabulary." Record words on the chalkboard or on large sheets of paper that can be put on the bulletin board. (Some teachers start such lists at beginning of term and keep them in evidence all year, adding new words as they are introduced.)

When teacher is satisfied that the list has been completed (she should suggest any that have been omitted) ask for volunteers to explain to the class the meaning of each term, demonstrating with concrete materials, diagrams, exercises on board, etc. These demonstrations will be far more valuable than mere verbalizations. Much recall will have taken place during this discussion of vocabulary.

Have children read and work the first eight Exercise Sets independently. Evaluate each Exercise Set carefully to determine what reteaching of concepts and skills is necessary.

Chapter 6

DIVISION OF RATIONAL NUMBERS

SOME FACTS YOU KNOW ABOUT RATIONAL NUMBERS

The whole numbers are useful in answering questions about how many objects are in a set of objects; but to answer questions about the parts of a single object, or a part of a set of objects, numbers of a new kind are used. The new set of numbers is called the set of rational numbers.

Numbers named by numerals like 0, 8, 0.9, $\frac{7}{3}$, $4\frac{2}{5}$, and $\frac{1}{6}$ are rational numbers of arithmetic.

Some rational numbers such as 0, 14, 198, and 7033 are also whole numbers.

A fraction is one kind of symbol, or name, for a rational number. Fractions we have used in our work so far are written with two numerals separated by a bar. The number named by the numeral below the bar is called the denominator and must be a counting number. The number named above the bar is called the numerator and must be a whole number.

Every rational number has many fraction names. Whole numbers may be named by fractions whose denominators are 1.

$$0,\ 1,\ 2,\ 3,\ \ldots$$

$$\frac{0}{1},\ \frac{1}{1},\ \frac{2}{1},\ \frac{3}{1},\ \ldots$$

Whole numbers also may be named by fractions having denominators other than one.

$$\frac{12}{6} = 2, \ \frac{24}{3} = 8, \ \frac{8}{2} = 4, \ \frac{16}{4} = 4$$

Notice that whenever a whole number is named by a fraction, the numerator is always a multiple of the denominator.

Rational numbers like $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{1}{8}$ also have many fraction names. Any rational number named by a fraction may be renamed by multiplying both the numerator and denominator by any counting number greater than one. Why do we say, "greater than 1"?

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} \ \ldots$$

$$\frac{4}{5} = \frac{8}{10} = \frac{12}{15} = \frac{16}{20} = \frac{20}{25} \ \ldots$$

$$\frac{7}{8} = \frac{14}{16} = \frac{21}{24} = \frac{28}{32} = \frac{35}{40} \ \ldots$$

Can you think of this way to rename a rational number named by a fraction in a different way? Could we think of it as multiplying the number by 1, and naming 1 by a fraction with denominator greater than 1?

$$\frac{1}{2} \times \frac{2}{2} = \frac{2}{4}$$

$$\frac{1}{2} \times \frac{3}{3} = \frac{3}{6}$$

$$\frac{1}{2} \times \frac{4}{4} = \frac{4}{8}$$

The simplest fraction name for a rational number is the one in which the numerator and denominator have no common factor except 1.

$$\frac{2}{3} = \frac{2 \times 1}{3 \times 1}$$

Numbers named by fractions like $\frac{6}{12}$, $\frac{10}{15}$, and $\frac{15}{18}$ can be renamed in simplest form by dividing the numerator and denominator by their greatest common factor.

$$\frac{6}{12} = \frac{6 \div 6}{12 \div 6} = \frac{1}{2}$$

$$\frac{10}{15} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3}$$

$$\frac{15}{18} = \frac{15 \div 3}{18 \div 3} = \frac{5}{6}$$

$$\frac{385}{66} = \frac{5 \times 7 \times 11}{2 \times 3 \times 11} = \frac{5 \times 7}{2 \times 3} = \frac{35}{6}$$

Names for rational numbers greater than 1 may be written in a variety of mixed forms.

$$3 + \frac{3}{4}, \quad 2\frac{11}{8}, \quad 1\frac{19}{8}, \quad 3\frac{2}{5}$$

Mixed forms such as $3\frac{2}{5}$, $6\frac{3}{4}$, and $7\frac{5}{8}$ are each a simplest mixed form because

(1) the numerator of the fraction is smaller than the denominator, and

(2) the fraction is in simplest form.

Exercise Set 1

1. Replace n in each sentence by a numeral to make a true statement.

$4\frac{7}{8} = \frac{n}{8}$ (39)

$7\frac{5}{16} = \frac{n}{16}$ (117)

$46\frac{1}{4} = \frac{n}{4}$ (185)

$9\frac{1}{2} = \frac{n}{2}$ (19)

$13\frac{2}{3} = \frac{n}{3}$ (41)

$20\frac{7}{8} = \frac{n}{8}$ (167)

2. Write the following in the simplest mixed form.

$\frac{9}{6} =$ ($1\frac{1}{2}$)

$5\frac{26}{6} =$ ($9\frac{1}{3}$)

$\frac{18}{3} =$ (6)

$\frac{34}{8} =$ ($4\frac{1}{4}$)

$7\frac{6}{3} =$ (9)

$3\frac{12}{8} =$ ($4\frac{1}{2}$)

3. Replace n in each of the following to make the sentence true.

$5\frac{2}{3} = 4\frac{n}{3}$ (5)

$8\frac{5}{6} = 7\frac{n}{6}$ (11)

$57\frac{4}{9} = 56\frac{n}{9}$ (13)

$10\frac{3}{4} = 9\frac{n}{4}$ (7)

$14\frac{7}{8} = 13\frac{n}{8}$ (15)

$21\frac{9}{10} = 20\frac{n}{10}$ (19)

4. Write three more members for each set below.

Set A = $\left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \left(\frac{4}{1}\right), \left(\frac{5}{1}\right), \left(\frac{6}{1}\right)\right\}$ (etc.)

Set B = $\left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \left(\frac{4}{4}\right), \left(\frac{5}{5}\right), \left(\frac{6}{6}\right)\right\}$ (etc.)

Set C = $\left\{\frac{3}{4}, \frac{6}{8}, \frac{12}{16}, \left(\frac{9}{12}\right), \left(\frac{15}{20}\right), \left(\frac{30}{40}\right)\right\}$ (etc.)

Set D = $\left\{\frac{2}{1}, \frac{4}{2}, \frac{10}{5}, \left(\frac{6}{3}\right), \left(\frac{8}{4}\right), \left(\frac{12}{6}\right)\right\}$ (etc.)

5. Find the simplest fraction name for each rational number.

a. $\frac{9}{15}$ $\left(\frac{3}{5}\right)$
b. $\frac{6}{18}$ $\left(\frac{1}{3}\right)$
c. $\frac{3}{27}$ $\left(\frac{1}{9}\right)$
d. $\frac{21}{24}$ $\left(\frac{7}{8}\right)$
e. $\frac{12}{14}$ $\left(\frac{6}{7}\right)$
f. $\frac{8}{32}$ $\left(\frac{1}{4}\right)$
g. $\frac{56}{63}$ $\left(\frac{8}{9}\right)$
h. $\frac{28}{49}$ $\left(\frac{4}{7}\right)$
i. $\frac{27}{36}$ $\left(\frac{3}{4}\right)$
j. $\frac{24}{40}$ $\left(\frac{3}{5}\right)$
k. $\frac{12}{16}$ $\left(\frac{3}{4}\right)$
l. $\frac{8}{10}$ $\left(\frac{4}{5}\right)$

6. Copy and supply the missing numerator or denominator.

a. $\frac{1}{2} = \frac{(8)}{16}$
b. $\frac{3}{4} = \frac{(9)}{12}$
c. $\frac{7}{8} = \frac{21}{(24)}$
d. $\frac{5}{6} = \frac{(25)}{30}$
e. $\frac{2}{3} = \frac{10}{(15)}$
f. $1\frac{4}{5} = 1\frac{(8)}{10}$
g. $\frac{5}{8} = \frac{25}{(40)}$
h. $\frac{1}{3} = \frac{(3)}{9}$

COMMON DENOMINATOR

In your work with rational numbers, it is often convenient to work with fractions whose denominators are the same. If two or more fractions have the same denominator, we say they have a common denominator. Common denominators may be found for any two fractions by finding common multiples of both denominators.

Often your knowledge of the multiplication facts is all you need to do this. For greater, less familiar denominators you can find the least multiple common to both denominators.

Consider fractions with denominators 15 and 21. Suppose Set F is the set of multiples of 15, and Set T is the set of multiples of 21. Can you list every member of the set? How do you indicate that there are more members than it is possible to list?

Set $F = \{15, 30, 45, 60, 75, 90, 105, \ldots\}$

Set $T = \{21, 42, 63, 84, 105, \ldots\}$

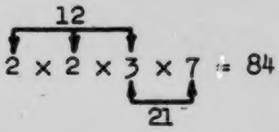
$F \cap T = \{105, \ldots\}$

Find the next seven members of Set F; of Set T. Find the next member of $F \cap T$.

Since 105 is the smallest number in Set $F \cap T$, it is called the least common multiple of 15 and 21. Any multiple of 105 is also a common multiple of 15 and 21. The product of 15 and 21, or of any two numbers, is always a common multiple of the numbers, but not always the least common multiple.

Complete factorization, or expressing a number as a product of primes, is the simplest method of finding a least common multiple for two numbers whose multiples you do not know.

Suppose the numbers are 12 and 21. Do you recall this convenient diagram you used in finding the least common multiple?



84 is the least common multiple of 21 and 12. 84 is also the least common denominator of fractions with denominators 12 and 21.

Exercise Set 2

1. Rename the numbers named by each pair of fractions below by fractions which have the least common denominator. You should use just multiplication facts for these.

   a. $\frac{1}{6}$, $\frac{7}{9}$ $(\frac{3}{18}, \frac{14}{18})$

   b. $\frac{5}{6}$, $\frac{2}{7}$ $(\frac{35}{42}, \frac{12}{42})$

   c. $\frac{3}{4}$, $\frac{7}{8}$ $(\frac{6}{8}, \frac{7}{8})$

   d. $\frac{3}{4}$, $\frac{5}{9}$ $(\frac{27}{36}, \frac{20}{36})$

   e. $\frac{2}{9}$, $\frac{4}{7}$ $(\frac{14}{63}, \frac{36}{63})$

   f. $\frac{6}{7}$, $\frac{3}{4}$ $(\frac{24}{28}, \frac{21}{28})$

2. Which is the greater rational number?

   a. $\frac{3}{10}$ or $\frac{1}{3}$ $(\frac{1}{3})$

   b. $\frac{5}{9}$ or $\frac{4}{7}$ $(\frac{4}{7})$

   c. $\frac{2}{3}$ or $\frac{4}{7}$ $(\frac{2}{3})$

   d. $\frac{3}{4}$ or $\frac{2}{5}$ $(\frac{3}{4})$

3. Arrange in order from least to greatest.

   $\frac{2}{3}$, $\frac{5}{6}$, $\frac{1}{2}$, $\frac{4}{9}$ $(\frac{4}{9}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6})$

4. Write the first five members of the set of multiples for each number. Underline the least common multiple for each pair. Then write the first two members of the set of common multiples for each pair.

   a. 16, 12
   $\{16, 32, \underline{48}, 64, 80\}$
   $\{12, 24, 36, \underline{48}, 60\}$
   $\{48, 96$

   b. 20, 25
   $\{20, 40, 60, 80, \underline{100}\}$
   $\{25, 50, 75, \underline{100}, 125\}$
   $\{100, 200$

   c. 13, 26
   $\{13, 26, 39, \underline{52}, 65\}$
   $\{26, \underline{52}, 78, 104, 130\}$
   $\{52, 104$

   d. 15, 5
   $\{\underline{15}, 30, 45, 60, 75\}$
   $\{5, 10, \underline{15}, 20, 25\}$
   $\{15, 30$

5. Name each pair of numbers by fractions with the least common denominator.

a. $\frac{5}{16}$, $\frac{7}{12}$ $(\frac{15}{48}, \frac{28}{48})$

b. $\frac{31}{20}$, $\frac{9}{25}$ $(\frac{155}{100}, \frac{36}{100})$

c. $\frac{8}{13}$, $\frac{7}{26}$ $(\frac{16}{26}, \frac{7}{26})$

d. $\frac{18}{15}$, $\frac{9}{5}$ $(\frac{18}{15}, \frac{27}{15})$

6. Find the least common multiple of each pair of numbers by using complete factorization.

a. 10, 14 (70)

b. 12, 20 (60)

c. 21, 30 (210)

d. 5, 16 (80)

e. 12, 14 (84)

f. 13, 15 (195)

7. Which is the greater rational number?

a. $\frac{5}{14}$ or $\frac{7}{12}$ $(\frac{7}{12})$

b. $\frac{8}{15}$ or $\frac{7}{16}$ $(\frac{8}{15})$

c. $\frac{11}{12}$ or $\frac{31}{32}$ $(\frac{31}{32})$

d. $\frac{9}{10}$ or $\frac{13}{14}$ $(\frac{13}{14})$

DECIMAL NAMES FOR RATIONAL NUMBERS

If you heard the words "three-tenths", you could write the fraction name $\frac{3}{10}$ or the decimal name 0.3.

Rational numbers whose fraction names have denominators 10, 100, or 1000 may be written directly as decimals.

$\frac{7}{10} = 0.7$ $\frac{39}{10} = 3.9$ $\frac{4325}{1000} = 4.325$

$\frac{2}{100} = 0.02$ $\frac{25}{1000} = 0.025$ $\frac{510}{100} = 5.10$

Some numbers like $\frac{1}{2}$ and $\frac{3}{4}$ must be renamed before they can be named by decimals.

$\frac{1}{2} = \frac{5}{10} = 0.5$ $\frac{3}{4} = \frac{75}{100} = 0.75$ $\frac{5}{8} = \frac{625}{1000} = 0.625$

Any decimal name for a rational number may be written directly as a fraction or mixed form.

$1.4 = 1\frac{4}{10}$ $2.007 = 2\frac{7}{1000}$

$0.59 = \frac{59}{100}$ $0.9 = \frac{9}{10}$

Recall that you can find a fraction name for a number if you know a decimal name for it, in this way:

The digits in the decimal indicate the numerator.

The place value of the last digit on the right indicates the denominator.

For example, $3.457 = \frac{3457}{1000}$ $.0025 = \frac{25}{10,000}$

The fraction obtained in this way is called the fraction form of the decimal.

What is the fraction form of 1.4? $\left(\frac{14}{10}\right)$ ^ of 2.007? $\left(\frac{2007}{1000}\right)$ ^ of 0.59? $\left(\frac{59}{100}\right)$

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Exercise Set 3

1. Name in words the rational number named by each decimal below.

a. 1.2	b. 0.07	c. 0.435
(one and two-tenths)	(seven hundredths	(four hundred thirty-five thousandths)

2. Complete the following chart.

Fraction Name	Numerator	Denominator	Decimal Name
a. $\frac{5}{100}$	5	100	0.05
b. $\frac{76}{100}$	(76)	(100)	(0.76)
c. $\frac{831}{1000}$	(831)	(1000)	(0.831)
d. $\frac{2408}{100}$	(2408)	(100)	(24.08)
e. $\frac{3012}{1000}$	(3012)	(1000)	(3.012)
f. $\frac{134}{10}$	(134)	(10)	(13.4)
g. $\frac{9}{1000}$	(9)	(1000)	(.009)
h. $\frac{234}{100}$	(234)	(100)	(2.34)

3. Match each fraction name in Column A with the decimal in Column B naming the same rational number.

Column A		Column B
a. $\frac{30}{100}$	(h.)	g. 3.0
b. $\frac{3}{100}$	(j.)	h. 0.30
c. $\frac{3}{1000}$	(i)	i. 0.003
d. $\frac{30}{10}$	(g)	j. 0.03
e. $\frac{30}{1000}$	(k.)	k. 0.030
f. $\frac{300}{10}$	(l)	l. 30.0

4. Complete the chart below.

Decimal Name of Rational Number	Numerator of Fraction Form	Denominator of Fraction Form	Fraction Form
a. 0.42	42	100	$\frac{42}{100}$
b. 0.056	(56)	(1000)	($\frac{56}{1000}$)
c. 7.3	(73)	(10)	($\frac{73}{10}$)
d. 2.08	(208)	(100)	($\frac{208}{100}$)
e. 11.01	(1101)	(100)	($\frac{1101}{100}$)
f. 0.9	(9)	(10)	($\frac{9}{10}$)
g. 3.405	(3405)	(1000)	($\frac{3405}{1000}$)
h. 4.071	(4071)	(1000)	($\frac{4071}{1000}$)
i. 2.06	(206)	(100)	$\frac{206}{100}$
j. 8.9	(89)	(10)	($\frac{89}{10}$)

5. Tell whether each of the rational numbers named below

> 1, < 1, or $= 1$

a. 0.01 (<)	d. 1.1 (>)	g. 0.999 (<)
b. 0.1 (<)	e. 0.001 (<)	h. 1.010 (>)
c. 0.9 (<)	f. 1.0 (=)	i. 0.901 (<)

6. In each of the following, write a decimal name for a rational number, such that

a. $m > 0.11$ and $m < 0.2$ → (.12, .13, .14, .15, .16,....) (Is more than one answer correct?)

b. $p > 0.009$ and $p < 0.009$ (No such number)

c. $t > 0.1$ and $t < 1.0$ (.2, .3, .4, .5, .6, ...)

d. $s > 1.1$ and $s < 1.9$ (1.2, 1.3, 1.4, 1.5, 1.6, ...)

7. Arrange in order from least to greatest.

$0.25, \quad 2\frac{1}{4}, \quad 1, \quad 0.02, \quad 1.02, \quad 2.002, \quad 2.2$

$(0.02,\ 0.25,\ 1,\ 1.02,\ 2.002,\ 2.2,\ 2\frac{1}{4})$

---

8. Write in expanded notation:

a. $4.9 \quad [(4 \times 1) + (9 \times \frac{1}{10})]$

b. $927.872 \quad [(9 \times 10^2) + (2 \times 10) + (7 \times 1) + (8 \times \frac{1}{10}) + (7 \times \frac{1}{10^2}) + (2 \times \frac{1}{10^3})]$

c. $56.63 \quad [(5 \times 10) + (6 \times 1) + (6 \times \frac{1}{10}) + (3 \times \frac{1}{10^2})]$

9. Write the decimal numeral for:

a. $\frac{45}{1} + \frac{7}{10} + \frac{5}{100} + \frac{3}{1000} \quad (45.753)$

b. $(7 \times 10^2) + (2 \times 10^1) + (4 \times 1) + (9 \times \frac{1}{10^1}) + (7 \times \frac{1}{10^2})$

(724.97)

Exercise Set 4

1. Which of these fractions may be written directly as decimals? Write the decimals if you can write them directly.

a. $\frac{5}{20}$ c. $\frac{30}{60}$ e. $\frac{3}{4}$

b. $\frac{465}{100}$ (4.65) d. $\frac{65}{10}$ (6.5) f. $\frac{9}{1000}$ (.009)

2. Rename each number so that it may be written directly as a decimal.

	Fraction Name	Multiply by	New Fraction Name	Decimal Name
a.	$\frac{12}{25}$	$\frac{4}{4}$	$\frac{48}{100}$	0.48
b.	$\frac{130}{500}$	$(\frac{2}{2})$	$(\frac{260}{1000})$	(0.260)
c.	$\frac{3}{2}$	$(\frac{5}{5})$	$(\frac{15}{10})$	(1.5)
d.	$\frac{4}{5}$	$(\frac{2}{2})$	$(\frac{8}{10})$	(0.8)
e.	$\frac{17}{20}$	$(\frac{5}{5})$	$(\frac{85}{100})$	(0.85)
f.	$\frac{125}{250}$	$(\frac{4}{4})$	$(\frac{500}{1000})$	(0.500)
g.	$\frac{60}{50}$	$(\frac{2}{2})$	$(\frac{120}{100})$	(1.20)
h.	$\frac{212}{200}$	$(\frac{5}{5})$	$(\frac{1060}{1000})$	(1.060)
i.	$\frac{2}{125}$	$(\frac{8}{8})$	$(\frac{16}{1000})$	(.016)

3. Which of the following numbers do not have a fraction name with denominator 10, 100, or 1000. (b, d, e) Rename all others as decimals.

a. $\frac{6}{8}$ (0.75)

b. $\frac{40}{55}$

c. $\frac{100}{400}$ (0.25)

d. $\frac{150}{900}$

e. $\frac{3}{33}$

f. $\frac{8}{40}$ (0.2)

4. Complete

Rational Number	Fraction Name	Decimal Name
one-fourth	$\frac{1}{4}$	0.25
one-half	($\frac{1}{2}$)	(0.5)
three-fourths	($\frac{3}{4}$)	(0.75)
four-twentieths	($\frac{4}{20}$)	(0.2)
three-fifths	($\frac{3}{5}$)	(0.6)
nine-tenths	($\frac{9}{10}$)	(0.9)

5. Match each fraction name in Column A with the correct decimal name in Column B.

Column A	Column B
a. $\frac{20}{25}$ (i.)	h. 0.008
b. $\frac{2}{250}$ (h.)	i. 0.80
c. $\frac{22}{25}$ (j.)	j. 0.88
d. $\frac{2}{25}$ (n.)	k. 8.08
e. $\frac{22}{250}$ (m.)	l. 0.808
f. $\frac{202}{250}$ (l.)	m. 0.088
g. $\frac{202}{25}$ (k.)	n. 0.08

Exercise Set 5

1. a. Add 29.9 and 37.06 (66.96)

   b. Multiply $(\frac{2}{3})^2$ and $(\frac{3}{4})^2$ $(\frac{1}{4})$

   c. Subtract 11.58 from 40 (28.42)

   d. Add $35\frac{5}{7}$ and $17\frac{3}{5}$ $(53\frac{11}{35})$

   e. Multiply 0.2 and 0.3 (0.06)

   f. Subtract $9\frac{7}{8}$ from $14\frac{1}{6}$ $(4\frac{7}{24})$

Read the following carefully. Express the relationships in each problem in a mathematical sentence. Solve, and write your answer in a complete sentence. Rational numbers may be represented by fractions or decimals.

2. One basketball player had an average of twenty-seven and nine-tenths points per game. A second player had an average of twenty-five and one-half points per game. $(27.9 > 25.5;\ n = 27.9 - 25.5)$ Which player had the better average. (The player with the average of 27.9) By how many points was it better? (2.4)

3. In a class of 32 children, there are 14 boys and 18 girls. 6 of the 14 boys are Scouts. 10 of the 18 girls are Campfire Girls. What part of the boys are Scouts? $(\frac{6}{14})$ What part of the girls are Campfire Girls? $(\frac{10}{18})$ What part of the class are Scouts? $(\frac{6}{32})$ What part of the class are Campfire Girls? $(\frac{10}{32})$ Which group represents the greater part of the class, Scouts or Campfire Girls? (Campfire Girls)

4. In one part of the Amazon River Valley the average rainfall per month is twenty-one and four-tenths inches. What is the total amount of rainfall for the year? (256.8 inches)
(The total rainfall was 256.8 inches)
$(n = 12 \times 21.4)$

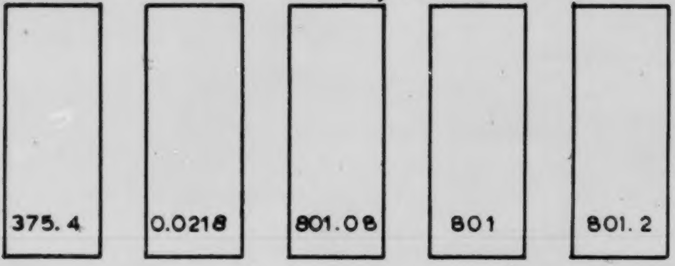
5. If a plane averages 560 miles an hour, how far will it travel in five and one-half hours? (3080 miles)
(It will travel 3080 miles)
$(n = 5\frac{1}{2} \times 560)$

6. On our vacation we averaged twenty-eight and seven-tenths miles per hour for an average of seven and one-half hours each day. What was the distance we traveled each day? (215.25 mi.)
(We traveled 215.25 miles)
$(n = 7\frac{1}{2} \times 28.7)$

7. The average house fuse will allow 15 amperes of electricity to pass through it. If you connect a two and five-tenths amp heater and a ten amp refrigerator to it, how many more amps can the fuse carry without burning out? (2.5 amps)
(It can carry 2.5 amps more)
$(2.5 + 10 + n = 15)$

8. A metric ton weighs two thousand two hundred four and six-tenths pounds. How many more pounds than the English ton is the metric ton? (204.6 lbs.)
(It weighs 204.6 lbs. more)
($n = 2204.6 - 2000$)

9. In the school library, books are arranged on the shelves just as numbers are arranged on a number line. Call numbers on books become greater as you move to the right. How should the following books be arranged on the shelf?
(They should be arranged: 0.0218, 375.4, 801, 801.08, 801.2)



10. The head is about $\frac{1}{5}$ of the height of a young boy. Jim is 60 inches tall. Will his head be above water when he stands in a pool marked 4 feet deep? (yes) Why?
(Because $\frac{4}{5}$ of 60 in. = 48 in = 4 ft.)
($n = \frac{4}{5} \times 60$; n should be > 48 inches)

P325

PROPERTIES OF RATIONAL NUMBERS

Exploration

Add any two whole numbers.

Try three more pairs.

Did you always have a whole number to use as a sum? (yes)

Can you think of any two whole numbers whose sum is not a whole number? (no)

Do you agree with this statement: (yes)

"The sum of any two whole numbers is always a whole number"?

1. Add the following pairs of rational numbers.

a. $1\frac{2}{3}$ + $\frac{1}{2}$ = $(2\frac{1}{6})$

b. $\frac{3}{4}$ + $\frac{1}{3}$ = $(1\frac{1}{12})$

c. $4\frac{5}{6}$ + $2\frac{3}{8}$ = $(7\frac{5}{24})$

d. $17\frac{1}{6}$ + $4\frac{5}{8}$ = $(21\frac{19}{24})$

e. $22\frac{3}{4}$ + $1\frac{1}{3}$ = $(24\frac{1}{12})$

f. $5\frac{5}{6}$ + $\frac{3}{4}$ = $(6\frac{7}{12})$

g. $58\frac{1}{2}$ + $2\frac{7}{10}$ = $(61\frac{1}{5})$

h. $32\frac{6}{7}$ + $17\frac{1}{2}$ = $(50\frac{5}{14})$

Was each sum a rational number? (yes)

Can you think of any two rational numbers whose sum is not a rational number? (no)

Do you agree that the sum of any two rational numbers is a rational number? (yes)

2. Perform the following operations.

a. 432 × 56 (24,192)

b. 708 × 9 (6,372)

c. 490 × 78 (38,220)

d. 5600 × 43 (240,800)

e. $3 \times \frac{5}{6}$ $(\frac{15}{6})$

f. $\frac{2}{3} \times \frac{4}{5}$ $(\frac{8}{15})$

g. $2 \times 3\frac{2}{5}$ $(6\frac{4}{5})$

h. $\frac{3}{4} \times 4\frac{1}{2}$ $(\frac{27}{8})$

When you multiplied two whole numbers, was the product always a whole number? (yes)

When you multiplied two rational numbers was the product always a rational number? (yes)

What generalization can you make about the product of two whole numbers? (It is a whole number) of two rational numbers? (It is a rational number)

3. Find the products of the following pairs of numbers.

a. $\frac{3}{4}$ and $\frac{4}{3}$ (1)

b. $1\frac{1}{2}$ and $\frac{2}{3}$ (1)

c. 5 and $\frac{1}{5}$ (1)

d. $\frac{1}{10}$ and 10 (1)

e. $1\frac{5}{6}$ and $\frac{6}{11}$ (1)

f. $\frac{7}{8}$ and $\frac{8}{7}$ (1)

What do you notice about the products you found? (They are all 1)

4. Can you think of any two whole numbers whose product is 1? (1 × 1 is the only one)
Did you think of 1 × 1? This is one more property of the interesting number 1:
When 1 is multiplied by 1, the product is 1.
No other whole number has this property.

Every rational number, however, with the exception of zero, does possess this property:

For any rational number, except zero, there is another rational number such that the product of the numbers is 1. Such numbers are called reciprocals of each other.

5. Look back at exercise 3.

What do you notice about the numbers in each example? (Each is the reciprocal of the other)

Name the reciprocal of each of the following:

a. $\frac{4}{1}$ ($\frac{1}{4}$) c. $\frac{3}{3}$ ($\frac{3}{3}$) e. $\frac{6}{15}$ ($\frac{15}{6}$)

b. $\frac{7}{9}$ ($\frac{9}{7}$) d. $\frac{2}{10}$ ($\frac{10}{2}$) f. 2 ($\frac{1}{2}$)

6. Which product expressions below are names for 1? (a, b, c)

a. $\frac{3}{4} \times \frac{4}{3}$ c. $5 \times \frac{1}{5}$ e. $1\frac{5}{6} \times \frac{4}{6}$

b. $1\frac{1}{2} \times \frac{2}{3}$ d. $\frac{1}{10} \times 1$ f. $\frac{2}{8} \times \frac{8}{7}$

7. Which of the following statements are true?

Explain your answer.

a. $7 + 4 = 4 + 7$ (True - commutative property)

b. $17.53 + 34.7 = 34.7 + 17.53$ (True - commutative property)

c. $3\frac{5}{8} + 12\frac{3}{4} = 12\frac{3}{4} + 3\frac{5}{8}$ (True - commutative property)

d. $1.5 - 0.3 = 0.3 - 1.5$ (False - $1.2 + 1.5 \neq 0.3$)

e. $(3.7 \times 2.4) \times 6.8 = 3.7 \times (2.4 \times 6.8)$ (True - associative property)

f. $(\frac{1}{2} \times \frac{3}{4}) \times 1\frac{2}{3} = \frac{1}{2} \times (\frac{3}{4} \times 1\frac{2}{3})$ (True - associative property)

g. $16 \div (8 \div 2) = (16 \div 8) \div 2$ (False - $16 \div 4 = 4$ and $2 \div 2 = 1$)

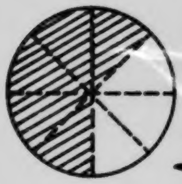
h. $2 \times (5 \times 3) = (2 \times 5) \times 3$ (True - associative property)

i. $7\frac{2}{5} - 3\frac{5}{6} = 3\frac{5}{6} - 7\frac{2}{5}$ (False. $7\frac{2}{5} - 3\frac{5}{6} = 3\frac{17}{30}$.
$3\frac{17}{30} = 3\frac{5}{6} - 7\frac{2}{5}$ is false because
$3\frac{17}{30} + 7\frac{2}{5} \neq 3\frac{5}{6}$

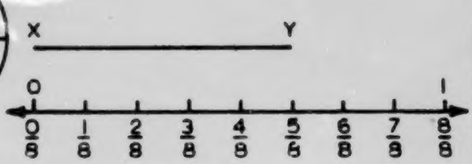
PICTURING RATIONAL NUMBERS

Rational numbers are used to describe the measure of a segment, region, or set in relation to a unit segment, unit region, or set.

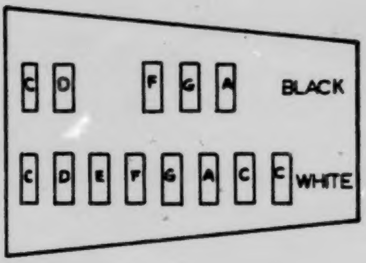
The following diagrams picture a region, a segment, and a set, each with measure $\frac{5}{8}$.



A



B



C

$\frac{5}{8}$ of the circular region in figure A is shaded.

The measure of $\overline{XY}$ in figure B is $\frac{5}{8}$.

$\frac{5}{8}$ of the set of white keys on the song bells have matching black keys.

Just as 5 is another name for

$$1 + 1 + 1 + 1 + 1$$

or

$$5 \times 1,$$

$\frac{5}{8}$ is another name for

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

or

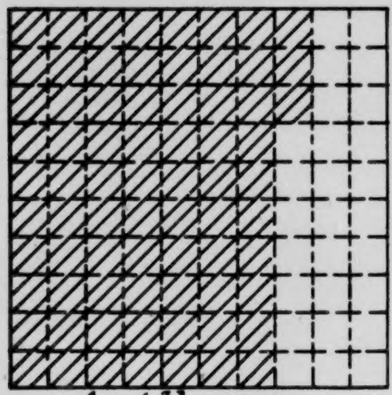
$$5 \times \frac{1}{8}$$

or

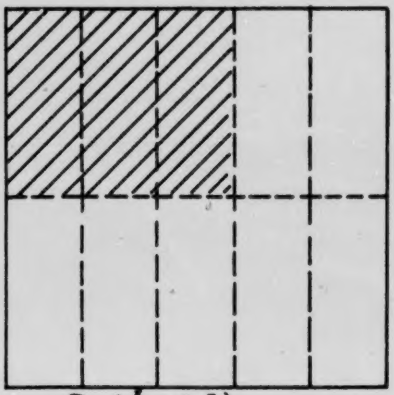
$$\frac{5}{8} \times 1.$$

Exercise Set 6

1. A and B are unit squares. Write both the fraction and decimal names that best describe the measure of each shaded region.

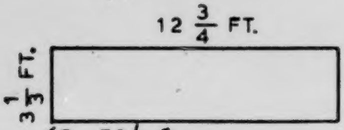


A ($\frac{73}{100}$, 0.73)



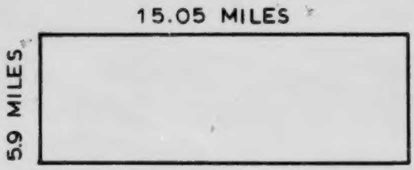
B ($\frac{3}{10}$, 0.3)

2. Find the perimeter and area of each rectangular region below.



$12\frac{3}{4}$ FT.
$3\frac{1}{3}$ FT.

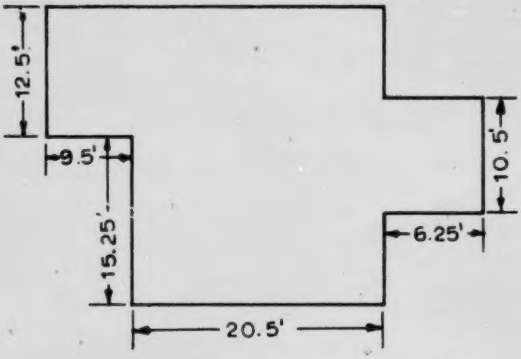
(P = $32\frac{1}{6}$ ft.
A = 42.5 sq. ft)



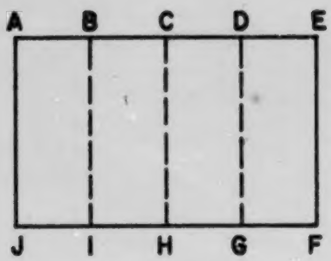
15.05 MILES
5.9 MILES

(P = 41.9 mi., A = 88.795 sq. mi.)

3. Find the area of the floor of the house whose floor plan is drawn below. (753.250 sq. ft.)



12.5'
9.5'
15.25'
20.5'
10.5'
6.25'



1. AEFJ is a region separated into 4 congruent regions.
   if m AEFJ = 1,

   a. m ABIJ = $\left(\frac{1}{4}\right)$ b. m ACHJ = $\left(\frac{1}{2}\right)$ c. m ADGJ = $\left(\frac{3}{4}\right)$

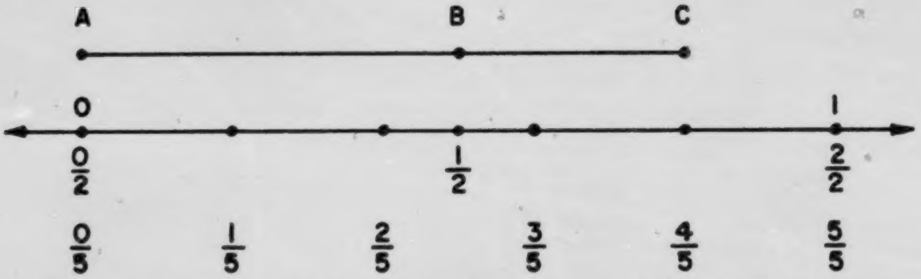
2. If m ADGJ = 1,

   a. m AEFJ = $\left(\frac{4}{3}\right)$ b. m ACHJ = $\left(\frac{2}{3}\right)$ c. m ABIJ = $\left(\frac{1}{3}\right)$

3. If m ABIJ = 1,

   a. m AEFJ = (4) b. m ADGJ = (3) c. m ACHJ = (2)

4. Write one addition and two subtraction sentences pictured by the diagram below. Write sentences using (a) fractions, and (b) decimals.

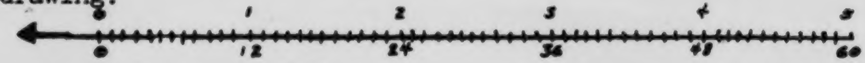


(a. $\frac{1}{2} + \frac{3}{10} = \frac{4}{5}$
0.5 + 0.3 = 0.8)

(b. $\frac{4}{5} - \frac{1}{2} = \frac{3}{10}$
0.8 − 0.5 = 0.3)

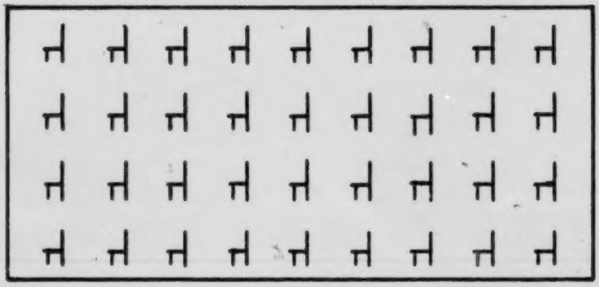
(c. $\frac{4}{5} - \frac{3}{10} = \frac{1}{2}$
0.8 − 0.3 = 0.5)

5. Illustrate the following problem with a number line drawing.



Tony made a chart to show the height in inches of all the children in his class. Allen told him he was $4\frac{2}{3}$ feet tall. How many inches did Tony record for Allen's height? (56 inches)

6. Below is a seating plan for a class of 36 children.



In how many other ways can you arrange the desks to make a series of equal rows? (8- see below for the possibilities and their fraction names) Draw a diagram similar to the one above to illustrate one arrangement and write the fraction name that best describes the measure of each row.

No. of chairs in row	no. of rows	Fraction name
4	9	$\frac{4}{36}$
18	2	$\frac{18}{36}$
2	18	$\frac{2}{36}$
12	3	$\frac{12}{36}$
3	12	$\frac{3}{36}$
36	1	$\frac{36}{36}$
1	36	$\frac{1}{36}$
6	6	$\frac{6}{36}$

Exercise Set 7

1. Complete the following mathematical sentences.

   a. $(4.9 \times 3.52) + (4.9 - 3.52) = \underline{(18.628)}$

   b. $(7 - 4\frac{5}{6}) + (5\frac{1}{3} \times 2\frac{1}{2}) = \underline{(15\frac{1}{2})}$

   c. $(\frac{8}{15} \times 1\frac{1}{2}) - \frac{1}{5} + \underline{(6\frac{1}{2})} = 7\frac{1}{10}$

   d. $(62.5 - 38.9) \times 4.3 = 25 + \underline{(76.48)}$

   e. $(1\frac{3}{4})^2 + (4 \times \frac{3}{7}) + \underline{(5\frac{25}{112})} = 10$

   f. $(0.3 \times 0.3) + (20.6 - 12.93) - \underline{(2.76)} = 5$

2. At one point, the river is 2.6 feet deep. At another, it is 5.1 feet deep. What is the difference in these two depths? (2.5 ft) (The difference in depths is 2.5 ft.)

3. It takes $3\frac{1}{3}$ seconds for a word sent by telegraph from one station to reach a second station, 4 seconds more to reach the third, and 3.9 seconds more to reach the fourth. How long does it take to transmit the word from the first to the fourth station? ($11\frac{7}{30}$ seconds) ($3\frac{1}{3} + 4 + 3.9 = n.$ $n = 11\frac{7}{30}$)

(It takes $11\frac{7}{30}$ seconds for it to reach the fourth station)

4. Ricky needed two 8.7 inch long axles and one 6 inch steering wheel pole for a toy racer he was making. He bought an iron rod 25 inches long. After he cut off the parts he needed, how many inches were left? (1.6 in.)
($2 \times 8.7 + 6 = n$
$25 - n = p$
$p = 1.6$) (There were 1.6 inches left.)

5. Plane A traveled 650 miles per hour for $3\frac{1}{2}$ hours. Plane B traveled 600 miles per hour for $3\frac{3}{4}$ hours. Which plane traveled the greater distance? (Plane A) How much greater was the distance? (25 mi.)
$(3\frac{1}{2} \times 650) - (3\frac{3}{4} \times 600) = 25$
(Plane A traveled 25 miles more than B)

6. What is the perimeter and what is the area of a rectangular room whose adjacent sides measure $24\frac{1}{2}$ feet by $16\frac{1}{2}$ feet?
(P. 82 ft., A = 404.25 sq. ft.)
$24\frac{1}{2} + 24\frac{1}{2} + 16\frac{1}{2} + 16\frac{1}{2} = P$ P. 82
$24\frac{1}{2} \times 16\frac{1}{2} = 404.25 = A$ A. 404.25)

(The perimeter is 82 ft. The area is 404.25 sq. ft.)

7. Discoverer II was in orbit around the earth for a period of 90.5 minutes. Discoverer I was in orbit 95.569 minutes. What is the difference in time of the two orbital flights? (5.069 minutes)

Exercise Set 8

Read the following problems carefully.

Express the relationships in each problem as a mathematical sentence.

1. Jane measured the tallest sunflower plant in her garden. It was 117 inches tall. What was its measurement in feet? ($n \times 12 = 117$: The plant was $9\frac{9}{12}$ or $9\frac{3}{4}$ ft. tall.)

2. Bill's height is $\frac{3}{4}$ of Tom's. Tom is 6 feet tall. How tall is Bill? ($n = \frac{3}{4} \times 6$: Bill is $4\frac{1}{2}$ ft. tall.)

3. Philip can throw a ball $\frac{4}{5}$ of the distance that Tim can. Tim can throw a ball 100 feet. How far can Philip throw it. ($n = \frac{4}{5} \times 100$: Philip can throw the ball 80 ft.)

4. A piece of garden hose 17 feet long is divided into 2 equal lengths. What is the measure of each piece? ($n \times 2 = 17$: The measure of each piece is $8\frac{1}{2}$ ft.)

5. Mary and Florence bicycled to a picnic. They left home at the same time and traveled the same distance. Mary reached the picnic in $\frac{3}{4}$ of an hour. Florence reached it in $\frac{1}{2}$ of an hour. Which girl took more time? How much more? ($\frac{3}{4} > \frac{1}{2}$; $n = \frac{3}{4} - \frac{1}{2}$: Mary took $\frac{1}{4}$ hr. more than Florence.)

6. Wildwood and Hillcrest Schools are having a debate. Each school has two speakers on its team. The debate will last one hour. The time will be divided equally among the speakers. What part of an hour will each speaker have?
($n \times 4 = 1$: Each speaker will have $\frac{1}{4}$ hr.)

7. The area of a rectangular flower bed is 24 square feet. The measure of one side is 8 feet. What is the measure of the adjacent side? ($n \times 8 = 24$: The measure of the other side is 3 ft.)

8. In January, 1961, 4.8 inches of rain fell in New Orleans. In February, 3.1 inches fell. How much rain did New Orleans have in the two months?
($n = 4.8 + 3.1$: New Orleans had 7.9 in. of rain in the two months.)

9. It takes Jupiter 11.862 earth years to orbit the sun. It takes Saturn 29.458 earth years to orbit the sun. What is the difference in earth years between the times required for Jupiter and Saturn to orbit the sun?
($n = 29.458 - 11.862$: The difference is 17.596 earth years.)

10. In 1959 a man-eating white shark measuring $16\frac{5}{6}$ feet was caught by rod and reel in Australia, setting a new record. A year later, a blue shark measuring $11\frac{1}{2}$ feet was caught by rod and reel in the United States, setting a world record also. How much greater was the length of the white shark?
($n = 16\frac{5}{6} - 11\frac{1}{2}$: The white shark was $5\frac{2}{6}$ or $5\frac{1}{3}$ ft. greater in length.)

II GETTING READY FOR DIVISION OF RATIONAL NUMBERS
DIVISION BY A RATIONAL NUMBER
COMPUTING QUOTIENTS OF RATIONAL NUMBERS

Objectives: To review the properties of rational numbers used to find a method for finding quotients of rational numbers.

To apply the properties of rational numbers in finding quotients.

To find a short method for computing quotients:

$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$, if $b \neq 0$, $c \neq 0$, $d \neq 0$

Vocabulary: Product, factor, unknown factor, dividend, divisor, quotient, divide ____ by ____, division sentence, multiplication sentence, quotient expression.

Materials: Strips of paper each about 8 inches by 2 inches. The teacher and each pupil should have several strips.

Suggested Teaching Procedure:

Division for whole numbers was defined as the inverse of multiplication; that is, $15 \div 3 = n$ means that n is a whole number such that $n \times 3 = 15$ and $3 \times n = 15$. With this definition, division was not always possible. For example, there is no whole number n for which this sentence is true:

$$17 \div 3 = n.$$

Division for rational numbers is defined in the same way:

$\frac{11}{9} \div \frac{4}{13} = n$ means that $n \times \frac{4}{13} = \frac{11}{9}$ and $\frac{4}{13} \times n = \frac{11}{9}$.

Before beginning the section, "Getting Ready for Division of Rational Numbers," you may wish to proceed as follows, using the strips of paper mentioned above.

You know how to add, subtract, and multiply rational numbers. Now we are going to study division of rational numbers.

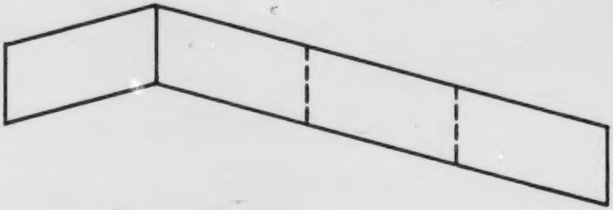
You have some strips of paper on your desk which you will use to diagram some mathematical sentences. The strips will help you picture division of rational numbers. Here are the mathematical sentences. Notice that in each, 1 is the product and a rational number is the known factor.

a. $1 \div \frac{3}{4} = n$ c. $1 \div \frac{1}{3} = n$ e. $1 \div \frac{2}{5} = n$

b. $1 \div \frac{2}{3} = n$ d. $1 \div \frac{3}{8} = n$ f. $1 \div \frac{3}{5} = n$

Consider that one whole strip represents a region whose measure is 1. How can you use one of the strips of paper to picture $1 \div \frac{3}{4}$? (Make as many small regions whose measure is $\frac{3}{4}$ as we can.)

How can you show a region whose measure is $\frac{3}{4}$? (Fold the strip into $\frac{4}{4}$. Use three of the fourths to picture three-fourths.)



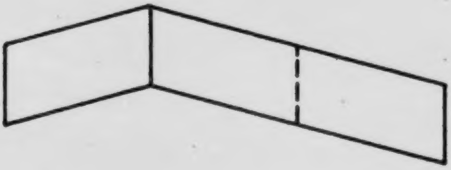
Strip 1

You have one piece whose measure is $\frac{3}{4}$ of the unit region. You have another piece which is smaller. What part of a $\frac{3}{4}$ piece is it? (It is $\frac{1}{3}$ of a $\frac{3}{4}$ piece.)

You can now tell what $1 \div \frac{3}{4}$ is. Look at your strip of paper. One whole strip is separated into two pieces. The larger piece has measure three-fourths. How does the smaller piece compare in size with the larger? (It is one-third as large.) The measure of the smaller piece is $\frac{1}{3}$ of three-fourths. What is $1 \div \frac{3}{4}$? ($1 \div \frac{3}{4} = 1\frac{1}{3}$, because you have one three-fourths piece and one-third of another such piece.)

Write $1 \div \frac{3}{4} = 1\frac{1}{3}$ on the chalkboard as the beginning of a summary of the day's discussion.

Use another strip of paper to represent a region whose measure is 1. Use it to find $1 \div \frac{2}{3}$. Describe how you do this. (Fold the strip into $\frac{3}{3}$. Use two of the thirds to make one piece with measure two-thirds; the other piece has measure $\frac{1}{2}$ of two-thirds. $1 \div \frac{2}{3} = 1\frac{1}{2}$.) (See Strip 2.)



Strip 2

The teacher should emphasize that a strip has a measure of 1 when the whole strip is the unit but it has a measure of $1\frac{1}{2}$ when the unit region is $\frac{2}{3}$ of a strip.

Have the mathematical sentences b through f solved in the same way.

Each solution should be added to the summary. The summary should have the form as shown at top of the next page.

a. $1 \div \frac{3}{4} = 1\frac{1}{3}$ d. $1 \div \frac{3}{8} = 2\frac{2}{3}$

b. $1 \div \frac{2}{3} = 1\frac{1}{2}$ e. $1 \div \frac{2}{5} = 2\frac{1}{2}$

c. $1 \div \frac{1}{3} = 3$ f. $1 \div \frac{3}{5} = 1\frac{2}{3}$

Look at your results in the problems above. The quotients are named by mixed forms. Rename them by fractions.

a. $1 \div \frac{3}{4} = \frac{4}{3}$ d. $1 \div \frac{3}{8} = \frac{8}{3}$

b. $1 \div \frac{2}{3} = \frac{3}{2}$ e. $1 \div \frac{2}{5} = \frac{5}{2}$

c. $1 \div \frac{1}{3} = \frac{3}{1}$ f. $1 \div \frac{3}{5} = \frac{5}{3}$

What do you notice about the results? (Each is the reciprocal of the known factor.)

Let's test another division with 1 as the product. Use a strip of paper to find $1 \div \frac{5}{16}$.

> The strip may be easily folded into halves, then into fourths, then into eighths, and then into sixteenths. Five sixteenths may then be torn off, another $\frac{5}{16}$ torn off, and another, and the $\frac{1}{16}$ left shown as $\frac{1}{5}$ of a $\frac{5}{16}$ piece.

What is $1 \div \frac{5}{16}$? $(3\frac{1}{5})$ We can write $1 \div \frac{5}{16} = \frac{16}{5}$. Is the result here the reciprocal of the known factor? (Yes) You have pictured division when the product is 1 and the known factor is a rational number. Later you will picture division when the product is a number different from 1.

> The development of the method presented in the text for computing n with fraction numerals is based on the definition of division and on these properties of rational numbers already known:
>
> 1. Multiplication Property. If b and d are not 0, then
>
> $$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

2. Reciprocal Property. For every rational number, except 0, there is another rational number such that the product of the two numbers is 1. If $a \neq 0$ and $b \neq 0$, the reciprocal of the number $\frac{a}{b}$ is $\frac{b}{a}$.

3. Associative Property.

$$\frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f}$$

4. Property of One. If n is a rational number, then $n \times 1 = 1 \times n = n$.

In addition to these four Properties, the idea of equality is important. The symbol " = " is taken to mean "is another name for." This is the meaning with which the pupils are familiar, but it has special importance in this development.

In the first Exploration for the section "Getting Ready to Divide by a Rational Number," division of rational numbers is defined as indicated above. The exercises in Set 9 are designed to focus attention on this definition and on Properties 2 and 4 in the list above.

The second Exploration focuses attention on the meaning of " = " and on the idea that the product of two rational numbers is a unique rational number. Thus, if "$5 \times n$" and "7" are names for the same number, "$2 \times (5 \times n)$" and "2×7" are also names for the same number.

In the section "Division by a Rational Number" the Properties are used to find the quotient of two rational numbers. No attempt should be made at this time to arrive at the usual "rule" for finding quotients. Emphasis should be on using the properties of rational numbers to find quotients of pairs of rational numbers.

You may wish to begin the discussion in this way:

Think of any two rational numbers. Suppose we use $\frac{8}{3}$ and $\frac{5}{8}$. You know how to find their sum. What is it? ($\frac{89}{24}$ or $3\frac{17}{24}$)

You know how to subtract $\frac{5}{8}$ from $\frac{8}{3}$. What is the other addend? ($\frac{49}{24}$ or $2\frac{1}{24}$)

You know how to find the product of $\frac{5}{8}$ and $\frac{8}{3}$. What is it? ($\frac{40}{24}$ or $\frac{5}{3}$)

Now we wish to find the quotient $\frac{8}{3} \div \frac{5}{8}$. Since division is related to multiplication, you can use some things you have learned about multiplication to find out how to divide.

Suppose that $\frac{8}{3} \div \frac{5}{8} = n$. (Write "$\frac{8}{3} \div \frac{5}{8} = n$" on the board) What is a multiplication sentence which states the same relationship as this division sentence? ($\frac{5}{8} \times n = \frac{8}{3}$) We know a number for $\frac{5}{8} \times n$. What is it? ($\frac{8}{3}$) You wish to find a number for n. What number times n equals n? ($1 \times n = n$) Now we have $\frac{5}{8} \times n$ and we want $1 \times n$. How could you operate on ($\frac{5}{8} \times n$) so as to have a product ($1 \times n$)? Could you multiply? (Yes) By what number? ($\frac{8}{5}$, because $\frac{8}{5} \times \frac{5}{8} = 1$) Try it. ($\frac{8}{5} \times (\frac{5}{8} \times n) = (\frac{8}{5} \times \frac{5}{8}) \times n = 1 \times n$) Now see what we have: (Write on board.)

$$\frac{5}{8} \times n = \frac{8}{3}$$

$$\frac{8}{5} \times (\frac{5}{8} \times n) = ?$$

What shall we write on the right side? Since ($\frac{5}{8} \times n$) and $\frac{8}{3}$ name the same number, $\frac{8}{5} \times (\frac{5}{8} \times n)$ and ($\frac{8}{5} \times \frac{8}{3}$) name the same number. So

$$\frac{8}{5} \times (\frac{5}{8} \times n) = \frac{8}{5} \times \frac{8}{3}.$$

How can we rewrite the left side?

$$(\frac{8}{5} \times \frac{5}{8}) \times n = \frac{8}{5} \times \frac{8}{3}.$$

What property was used? (Associative)

Rewrite the left side again. ($1 \times n = \frac{8}{5} \times \frac{8}{3}$) Rewrite it again. ($n = \frac{8}{5} \times \frac{8}{3}$) Now find a simpler name for the number named by the product expression. ($\frac{64}{15}$) So

$$n = \frac{64}{15} \quad \text{and}$$

$$\frac{8}{3} \div \frac{5}{8} = \frac{64}{15}.$$

Continue with other examples. Then work through the Exploration with the pupils. Emphasize the properties used. Exercise Set 10 can be used for individual work in finding quotients, following the models shown.

Here is a slightly different development which you could use instead of, or in addition to, the one above.

As before,

$$\frac{8}{3} \div \frac{5}{8} = n, \text{ so}$$

$$\frac{5}{8} \times n = \frac{8}{3}$$

What number must replace n to make a true statement?

Suppose n were 2? Then $\frac{5}{8} \times 2 = ?$

Suppose n were $\frac{9}{4}$? Then $\frac{5}{8} \times \frac{9}{4} = ?$

Finding n in this way would take a long time.

Suppose n were $\frac{8}{5}$. Then

$$\frac{5}{8} \times \frac{8}{5} = 1$$

To get a numeral for $\frac{8}{3}$, by what number must we multiply $(\frac{5}{8} \times \frac{8}{5})$? Since $1 \times \frac{8}{3} = \frac{8}{3}$, multiply by $\frac{8}{3}$.

$$(\frac{5}{8} \times \frac{8}{5}) \times \frac{8}{3} = 1 \times \frac{8}{3}.$$

The multiplication sentence with n was $\frac{5}{8} \times n = \frac{8}{3}$, so use the Associative Property.

$$\frac{5}{8} \times (\frac{8}{5} \times \frac{8}{3}) = \frac{8}{3}.$$

Rewrite this multiplication sentence as a division sentence.

$$\frac{8}{3} \div \frac{5}{8} = \frac{8}{5} \times \frac{8}{3}$$

So, $$\frac{8}{3} \div \frac{5}{8} = \frac{64}{15}.$$

In the section "Computing Quotients of Rational Numbers" the method for finding quotients developed in the previous section is examined to find a clue for a short method of computing quotients. Work through the Exploration with the class (or develop an example on the board). Stop short of simplifying the final product expression, centering attention on the facts that $\frac{7}{8} \div \frac{3}{4} = n$, and that application of the Properties shows that $n = \frac{7}{8} \times \frac{4}{3}$. Therefore,

$$\frac{7}{8} \div \frac{3}{4} = \frac{7}{8} \times \frac{4}{3}$$

When the pupils have learned to compute quotients using the relation

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c},$$

it is important to return to the longer methods of solution used in the beginning frequently enough to remind them that this convenient short cut is a consequence of the definitions and properties of rational numbers previously observed.

GETTING READY FOR DIVISION OF RATIONAL NUMBERS

Exploration

You have studied multiplication of rational numbers. Now you will study division of rational numbers.

You know that division with whole numbers is a process of finding an unknown factor of a product when one factor is known. We shall think of division with rational numbers as having the same meaning. That is, the sentence

$$\frac{5}{8} \div \frac{2}{3} = \frac{15}{16}$$

means $$\frac{2}{3} \times \frac{15}{16} = \frac{5}{8}$$

Is this multiplication sentence true?

Until now, you have used only factors which were counting numbers. In thinking about division of rational numbers we shall also consider rational numbers as factors of a product which is a rational number. For example, in the sentences

$$\frac{5}{8} \times n = 7 \qquad \text{or} \qquad 7 \div \frac{5}{8} = n$$

we shall speak of 7 as the product, $\frac{5}{8}$ as one factor, and n as the other factor. We also can use the language of division and call 7 the dividend, $\frac{5}{8}$ the divisor and n the quotient.

In the next set of exercises use what you have learned about products of rational numbers and about the operation of division to get ready to divide by a rational number.

Exercise Set 9

1. Find a fraction name for each product expression.

a. $\frac{2}{3} \times \frac{3}{2} = n$ $(\frac{6}{6})$

b. $\frac{5}{9} \times \frac{9}{5} = n$ $(\frac{45}{45})$

c. $8 \times \frac{1}{8} = n$ $(\frac{8}{8})$

d. $\frac{2}{15} \times 7\frac{1}{2} = n$ $(\frac{30}{30})$

e. $1\frac{1}{3} \times \frac{3}{4} = n$ $(\frac{12}{12})$

f. $\frac{10}{19} \times 1\frac{9}{10} = n$ $(\frac{190}{190})$

g. $\frac{237}{459} \times \frac{459}{237} = n$ $(\frac{108,78}{108,78}$

h. $\frac{57}{59} \times 1\frac{2}{57} = n$ $(\frac{3363}{3363})$

i. $3\frac{1}{7} \times \frac{7}{22} = n$ $(\frac{154}{154})$

2. In example 1, each product is the number (1). In each product expression, one factor is the (reciprocal) of the other.

3. For each sentence, find a number n which makes the sentence true.

a. $\frac{4}{3} \times n = 1$ $(\frac{3}{4})$

b. $n \times \frac{5}{8} = 1$ $(\frac{8}{5})$

c. $\frac{4}{15} \times n = \frac{4}{15}$ (1)

d. $\frac{1}{10} \times n = 1$ $(\frac{10}{1})$

e. $n \times \frac{1}{13} = 1$ $(\frac{13}{1})$

f. $n \times \frac{23}{8} = \frac{23}{8}$ (1)

g. $(\frac{10}{13} \times \frac{13}{10}) \times \frac{7}{8} = n$ $(\frac{7}{8})$

h. $(\frac{8}{9} \times \frac{9}{8}) \times n = \frac{7}{15}$ $(\frac{7}{15})$

i. $(\frac{2}{3} \times n) \times \frac{3}{5} = \frac{3}{5}$ $(\frac{3}{2})$

j. $(9 \times n) \times \frac{15}{7} = \frac{15}{7}$ $(\frac{1}{9})$

k. $(n \times \frac{1}{7}) \times \frac{5}{11} = \frac{5}{11}$ $(\frac{7}{1})$

l. $\frac{8}{9} \times (n \times \frac{3}{4}) = \frac{8}{9}$ $(\frac{4}{3})$

m. $\frac{7}{5} \times (\frac{9}{10} \times \frac{10}{9}) = n$ $(\frac{7}{5})$

n. $n \times (\frac{7}{11} \times \frac{11}{7}) = \frac{5}{13}$ $(\frac{5}{13})$

4. Here are some division sentences. Write the relationship in each sentence as a multiplication sentence.

a. $20 \div n = 4$
$(n \times 4 = 20)$ or $(4 \times n = 20)$
$(n = 5)$

b. $65 \div 13 = n$
$(n \times 13 = 65)$ etc.
$(n = 5)$

c. $42 \div 14 = n$
$(n \times 14 = 42)$
$(n = 3)$

d. $42 \div n = 14$
$(n \times 14 = 42)$
$(n = 3)$

e. $n \div 14 = 42$
$(42 \times 14 = n)$
$(n = 588)$

f. $19 \div n = 19$
$(n \times 19 = 19)$
$(n = 1)$

g. $1 \div n = 19$
$(n \times 19 = 1)$
$(n = \frac{1}{19})$

h. $1 \div 19 = n$
$(n \times 19 = 1)$
$n = \frac{1}{19})$

i. $n \div \frac{2}{3} = \frac{3}{2}$
$(\frac{3}{2} \times \frac{2}{3} = n)$
$(n = 1)$

5. For each sentence in exercise 4, find a number n which makes the sentence true. (Use your multiplication sentences if you prefer to do so.) (See above)

6. Rewrite each division sentence as a multiplication sentence which states the same relationship.

a. $15 \div n = (\frac{2}{3} \times \frac{3}{2})$
$[n \times (\frac{2}{3} \times \frac{3}{2}) = 15]$ $(n = 15)$

b. $15 \div (\frac{9}{7} \times \frac{7}{9}) = n$
$[n \times (\frac{9}{7} \times \frac{7}{9}) = 15]$ $(n = 15)$

c. $\frac{4}{9} \div (\frac{5}{8} \times \frac{8}{5}) = n$
$[n \times (\frac{5}{8} \times \frac{8}{5}) = \frac{4}{9}]$ $(n = \frac{4}{9})$

d. $n \div (\frac{7}{3} \times \frac{3}{7}) = \frac{2}{9}$
$[\frac{2}{9} \times (\frac{7}{3} \times \frac{3}{7}) = n]$ $(n = \frac{2}{9})$

e. $7\frac{1}{2} \div n = (\frac{11}{4} \times \frac{4}{11})$
$[n \times (\frac{11}{4} \times \frac{4}{11}) = 7\frac{1}{2}]$ $(n = 7\frac{1}{2})$

f. $1 \div \frac{13}{3} = n$
$[n \times \frac{13}{3} = 1]$ $(n = \frac{3}{13})$

g. $(\frac{10}{7} \times \frac{7}{10}) \div n = 2\frac{3}{4}$
$(n \times 2\frac{3}{4} = (\frac{10}{7} \times \frac{7}{10})$ $(n = \frac{4}{11})$

h. $6\frac{2}{3} \div (\frac{6}{7} \times \frac{7}{6}) = n$
$[n \times (\frac{6}{7} \times \frac{7}{6}) = 6\frac{2}{3}]$ $(n = 6\frac{2}{3})$

i. $n \div (\frac{7}{12} \times 1\frac{5}{7}) = 5\frac{1}{2}$
$[5\frac{1}{2} \times (\frac{7}{12} \times 1\frac{5}{7}) = n]$ $(n = 5\frac{1}{2})$

j. $n \div 8\frac{1}{2} = 2$
$(2 \times 8\frac{1}{2} = n)$ $(n = 17)$

k. $n \div \frac{7}{9} = \frac{2}{3}$
$(\frac{2}{3} \times \frac{7}{9} = n)$ $(n = \frac{14}{27})$

l. $\frac{8}{15} \div \frac{2}{3} = n$
$(n \times \frac{2}{3} = \frac{8}{15})$ $(n = \frac{4}{5})$

m. $(\frac{8}{15} \times \frac{3}{2}) \div n = (\frac{2}{3} \times \frac{3}{2})$
$[n \times (\frac{2}{3} \times \frac{3}{2}) = (\frac{8}{15} \times \frac{3}{2})]$
$(n = \frac{24}{30}$ or $\frac{4}{5})$

7. For each sentence you wrote in exercise 6, find the number n which makes your multiplication sentence true. (See problem 6)

8. For each multiplication sentence below, write two division sentences which state the same relationship.

a. $\frac{2}{3} \times \frac{3}{2} = 1$ d. $\frac{2}{15} \times 7\frac{1}{2} = 1$ g. $\frac{237}{459} \times \frac{459}{237} = 1$

b. $\frac{5}{9} \times \frac{9}{5} = 1$ e. $1\frac{1}{3} \times \frac{3}{4} = 1$ h. $\frac{57}{59} \times 1\frac{2}{57} = 1$

c. $8 \times \frac{1}{8} = 1$ f. $\frac{10}{19} \times 1\frac{9}{10} = 1$ i. $3\frac{1}{7} \times \frac{7}{22} = 1$

(a. $1 \div \frac{3}{2} = \frac{2}{3}$
a. $1 \div \frac{2}{3} = \frac{3}{2}$)

(d. $1 \div 7\frac{1}{2} = \frac{2}{15}$
d. $1 \div \frac{2}{15} = 7\frac{1}{2}$)

(g. $1 \div \frac{459}{237} = \frac{237}{459}$
g. $1 \div \frac{237}{459} = \frac{459}{237}$)

(b. $1 \div \frac{9}{5} = \frac{5}{9}$
b. $1 \div \frac{5}{9} = \frac{9}{5}$)

(e. $1 \div \frac{3}{4} = 1\frac{1}{3}$
e. $1 \div 1\frac{1}{3} = \frac{3}{4}$)

(h. $1 \div 1\frac{2}{57} = \frac{57}{59}$
h. $1 \div \frac{57}{59} = 1\frac{2}{57}$)

(c. $1 \div \frac{1}{8} = 8$
c. $1 \div 8 = \frac{1}{8}$)

(f. $1 \div 1\frac{9}{10} = \frac{10}{19}$
f. $1 \div \frac{10}{19} = 1\frac{9}{10}$)

(i. $1 \div \frac{7}{22} = 3\frac{1}{7}$
i. $1 \div 3\frac{1}{7} = \frac{7}{22}$)

9. When a product is the number 1, and one factor is a rational number, the unknown factor is the (reciprocal) of the first factor.

We write:

$$1 \div \frac{a}{b} = \frac{b}{a}, \text{ when } a \neq 0 \text{ and } b \neq 0.$$

Exploration

You know that $6 \times 5 = 30$. The " = " means that "6×5" and "30" are two names for the same number.

If $6 \times 5 = 30$, what number is

$2 \times 6 \times 5$? Is it 2×30? (yes)

The Associative Property tells us that

$$2 \times (6 \times 5) = (2 \times 6) \times 5.$$

So $2 \times 6 \times 5$ can be thought of as $2 \times (6 \times 5)$ or as $(2 \times 6) \times 5$. If we choose to think of it as $2 \times (6 \times 5)$, then

$$2 \times 6 \times 5 = 2 \times 30.$$

Now consider this sentence:

$$7 \times n = 42.$$

If $7 \times n$ and 42 are two names for the same number, you know that $n = 6$. No other number for n will make $7 \times n = 42$ a true sentence.

If $7 \times n = 42$, what about $5 \times (7 \times n)$? How must you operate on 42 to get a product which names the same number as $5 \times (7 \times n)$?

If $7 \times n = 42$,

then $5 \times 7 \times n = \underline{(5)} \times 42$

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Complete these sentences. Remember the meaning of " = ".

1. If $3 \times 8 = 24$
then $2 \times 3 \times 8 = \underline{(2)} \times 24$

2. If $48 = 6 \times 8$
then $\underline{(3)} \times 48 = 3 \times 6 \times 8$

3. If $21 = 7 \times 3$
then $21 \times \frac{3}{5} = 7 \times 3 \times \underline{(\frac{3}{5})}$

4. If $3 \times 15 = 45$
then $3 \times 15 \times \frac{5}{8} = 45 \times \underline{(\frac{5}{8})}$

5. If $6 \times n = 18$
then $2 \times 6 \times n = \underline{(2)} \times 18$

6. If $\frac{2}{3} \times r = 12$
then $\frac{3}{2} \times \frac{2}{3} \times r = \underline{(\frac{3}{2})} \times 12$

7. If $n \times \frac{3}{4} = 15$
then $n \times \frac{3}{4} \times \frac{6}{7} = \underline{(15)} \times \frac{6}{7}$

8. If $\frac{5}{7} \times n = 40$
then $\frac{5}{7} \times n \times \frac{7}{5} = 40 \times \underline{(\frac{7}{5})}$

9. If $36 = \frac{9}{10} \times k$
then $\frac{10}{9} \times 36 = \underline{\frac{10}{9}} \times \frac{9}{10} \times k$

10. If $\frac{2}{9} = \frac{4}{5} \times n$

then $\frac{9}{2} \times \frac{2}{9} = \frac{9}{2} \times \underline{(\frac{4}{5})} \times \underline{(n)}$

11. $\underline{(\frac{5}{3})} \times \frac{3}{5} \times n = 1 \times n$, if n is any rational number.

12. $\frac{2}{3} \times \underline{(\frac{3}{2})} \times n = 1 \times n$

13. $n \times \frac{19}{5} \times \underline{(\frac{5}{19})} = n \times 1$

14. $1 \times n = \frac{4}{5} \times \underline{(\frac{5}{4})} \times n$

15. $n \times 1 = n \times \frac{10}{9} \times \underline{(\frac{9}{10})}$

16. If $\frac{7}{8} \times n = 35$

then $\frac{8}{7} \times \frac{7}{8} \times n = \underline{(\frac{8}{7})} \times 35$

and $\underline{(1)} \times n = \underline{(\frac{8}{7})} \times 35$

and $n = \underline{(40)}$

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17. If $n \times \frac{3}{4} = \frac{15}{7}$

then $n \times \frac{3}{4} \times \frac{4}{3} = \frac{15}{7} \times \underline{(\frac{4}{3})}$

and $n \times 1 = \frac{15}{7} \times \underline{(\frac{4}{3})}$

and $n = \underline{\quad}$

18. If $n \times \frac{8}{5} = \frac{16}{7}$

then $n \times \frac{8}{5} \times \frac{5}{8} = \underline{(\frac{16}{7})} \times \frac{5}{8}$

and $n \times \underline{(1)} = \underline{(\frac{80}{56})}$

and $n = \underline{(\frac{80}{56}} \text{ or } \frac{10}{7})$

19. If $\frac{9}{2} = \frac{2}{7} \times n$

then $\frac{7}{2} \times \frac{9}{2} = \frac{7}{2} \times \underline{(\frac{2}{7})} \times n$

and $\frac{7}{2} \times \frac{9}{2} = \underline{(1)} \times n$

and $\underline{(\frac{63}{4})} = n$

20. If $\frac{8}{3} \times n = \frac{7}{12}$

then $\frac{3}{8} \times \frac{8}{3} \times n = \underline{(\frac{3}{8})} \times \frac{7}{12}$

and $\underline{(1)} \times n = \underline{(\frac{3}{8})} \times \frac{7}{12}$

and $n = \underline{(\frac{21}{96}} \text{ or } \frac{7}{32})$

DIVISION BY A RATIONAL NUMBER

Exploration

You know how to find an unknown factor of the number 1 when you know the other factor. Now consider finding an unknown factor of any rational number.

Think about this sentence:

$$\frac{7}{2} \div \frac{3}{4} = n$$

1. What multiplication sentence states the same relationship? $\left(\frac{7}{2} = \frac{3}{4} \times n\right)$

2. Complete this sentence:

$$\underline{\left(\frac{4}{3}\right)} \times \frac{3}{4} \times n = 1 \times n$$

Your answers to exercises 1 and 2 suggest a way to find $1 \times n$, or n.

Your multiplication sentence was

$$\frac{3}{4} \times n = \frac{7}{2}$$

You wish to find n, or $1 \times n$.

Your answer to exercise 2 shows that

$$\frac{4}{3} \times \frac{3}{4} \times n = 1 \times n$$

This suggests multiplying $\frac{3}{4} \times n$ by $\frac{4}{3}$.

3. Look at these sentences. Can you complete them?

a. $\frac{3}{4} \times n = \frac{7}{2}$

b. $\frac{4}{3} \times \frac{3}{4} \times n = \underline{(\frac{4}{3})} \times \frac{7}{2}$

c. $\underline{(1)} \times n = \underline{(\frac{4}{3})} \times \frac{7}{2}$

d. $n = \underline{(\frac{4}{3})} \times \frac{7}{2}$, or $\underline{(\frac{28}{6})}$,

The simplest fraction name is $\underline{(\frac{14}{3})}$.

e. So $\frac{7}{2} \div \frac{3}{4} = \underline{(\frac{14}{3})}$.

Now think about this example:

4. $\frac{5}{4} \div \frac{2}{7} = n$

Use the multiplication sentence

$$n \times \frac{2}{7} = \frac{5}{4}$$

a. You have a number for $n \times \frac{2}{7}$. You want to find $n \times 1$.

$$n \times 1 = n \times \frac{2}{7} \times \underline{(\frac{7}{2})}$$

b. Now use the result in exercise a.

Multiply $n \times \frac{2}{7}$ by $\underline{(\frac{7}{2})}$

c. $n \times \frac{2}{7} \times \underline{(\frac{7}{2})} = \frac{5}{4} \times \underline{(\frac{7}{2})}$

d. $n \times \underline{(1)} = \frac{5}{4} \times \underline{(\frac{7}{2})}$

e. $n = \underline{(\frac{35}{8})}$

f. $\frac{5}{4} \div \frac{2}{7} = \underline{(\frac{35}{8})}$

Now look at this example.

$$3\frac{1}{3} + 1\frac{3}{8} = n$$

Rename the numbers: $\frac{10}{3} + \frac{11}{8} = n$

State the multiplication sentence:

$$n \times \frac{11}{8} = \frac{10}{3}$$

To get $n \times 1$, multiply $(n \times \frac{11}{8})$ by $\frac{8}{11}$:

$$(n \times \frac{11}{8}) \times \frac{8}{11} = \frac{10}{3} \times \frac{8}{11}$$

Use the Associative Property:

$$n \times (\frac{11}{8} \times \frac{8}{11}) = \frac{10}{3} \times \frac{8}{11}$$

$\frac{11}{8} \times \frac{8}{11} = 1$

$$n \times 1 = \frac{10}{3} \times \frac{8}{11}$$

Use the Property of One: $n = \frac{10}{3} \times \frac{8}{11}$

Now you have a product expression:

$$n = \frac{10 \times 8}{3 \times 11} = \frac{80}{33} = \frac{66}{33} + \frac{14}{33}$$

$$n = 2\frac{14}{33}$$

So $3\frac{1}{3} + 1\frac{3}{8} = 2\frac{14}{33}$

Check: Does $1\frac{3}{8} \times 2\frac{14}{33} = 3\frac{1}{3}$?

$$\frac{11}{8} \times \frac{80}{33} = \frac{11 \times 80}{8 \times 33} = \frac{11 \times 8 \times 10}{8 \times 11 \times 3} = \frac{10}{3} = 3\frac{1}{3}$$

Now try this example: Write out your work as shown above.

$$\frac{16}{5} + \frac{8}{3} = n$$

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Exercise Set 10

Copy the work for these division examples and complete each sentence to make it true.

1. $\frac{7}{8} \div \frac{2}{3} = n$

$n \times \frac{2}{3} = \frac{7}{8}$

$(n \times \frac{2}{3}) \times \underline{(\frac{3}{2})} = \frac{7}{8} \times \underline{(\frac{3}{2})}$ (same number in both blanks)

$n \times (\frac{2}{3} \times \underline{(\frac{3}{2})}) = \frac{7}{8} \times \underline{(\frac{3}{2})}$ (same number as before)

$n \times 1 = \frac{7}{8} \times \underline{(\frac{3}{2})}$ (same number as before)

$n = \frac{7}{8} \times \underline{(\frac{3}{2})}$

$n = \underline{(\frac{21}{16}}\text{ or } 1\frac{5}{16})$ (simplify product expression)

$\frac{7}{8} \div \frac{2}{3} = \underline{(1\frac{5}{16})}$

2. $\frac{15}{4} \div n = \frac{5}{2}$

$n \times \frac{5}{2} = \frac{15}{4}$

$(n \times \frac{5}{2}) \times \underline{(\frac{2}{3})} = \frac{15}{4} \times \underline{(\frac{2}{3})}$ (Same number in both blanks)

$n \times (\frac{5}{2} \times \underline{(\frac{2}{3})}) = \frac{15}{4} \times \underline{(\frac{2}{3})}$

$n \times \underline{(1)} = \frac{15}{4} \times \underline{(\frac{2}{3})}$ (Different numbers)

$n = \frac{15}{4} \times \underline{(\frac{2}{3})}$

$n = \underline{(\frac{30}{20}}\text{ or } \frac{3}{2})$

$\frac{15}{4} \div \underline{(\frac{3}{2})} = \frac{5}{2}$

3. Are the numbers suggested for n correct? Check by multiplying.

a. $\frac{6}{7} \div \frac{8}{5} = n$

$n = \frac{15}{28}$ (yes: $\frac{15}{28} \times \frac{8}{5} = \frac{6}{7}$)

b. $3\frac{5}{7} \div n = 1\frac{3}{5}$

$n = 2\frac{1}{4}$ (no: $2\frac{1}{4} \times 1\frac{3}{5} \neq 3\frac{5}{7}$)

Find a number n which makes each sentence true.

Write your work as shown in exercises 1 and 2.

4. $9\frac{3}{4} \div \frac{13}{7} = n$ $(5\frac{1}{4}) \rightarrow$

$$9\frac{3}{4} \div \frac{13}{7} = n$$
$$n \times \frac{13}{7} = \frac{39}{4}$$
$$\left(n \times \frac{13}{7}\right) \times \frac{7}{13} = \frac{39}{4} \times \frac{7}{13}$$
$$n \times \left(\frac{13}{7} \times \frac{7}{13}\right) = \frac{39}{4} \times \frac{7}{13}$$
$$n \times 1 = \frac{39 \times 7}{4 \times 13}$$
$$n = \frac{13 \times 3 \times 7}{4 \times 13} = \frac{21}{4}$$
$$9\frac{3}{4} \div \frac{13}{7} = \frac{21}{7}$$

5. $2\frac{5}{6} \div n = 5\frac{1}{2}$ $(\frac{17}{33})$

6. $1\frac{3}{4} \div 3\frac{1}{2} = n$ $(\frac{1}{2})$

7. $2\frac{3}{10} \div n = \frac{4}{5}$ $(2\frac{7}{8})$

COMPUTING QUOTIENTS OF RATIONAL NUMBERS

Exploration

You have learned how to divide by a rational number, using the meaning of division and the properties of multiplication of rational numbers. Now see whether there is a short way to compute quotients.

1. Explain lines b to f of this example.

a. $\frac{7}{8} \div \frac{3}{4} = n$

b. $n \times \frac{3}{4} = \frac{7}{8}$ (Division sentence rewritten as multiplication sentence.)

c. $(n \times \frac{3}{4}) \times \frac{4}{3} = \frac{7}{8} \times \frac{4}{3}$ (Multiply $(n \times \frac{3}{4})$ and $\frac{7}{8}$ by $\frac{4}{3}$)

d. $n \times (\frac{3}{4} \times \frac{4}{3}) = \frac{7}{8} \times \frac{4}{3}$ (Use Associative Property)

e. $n \times 1 = \frac{7}{8} \times \frac{4}{3}$ ($\frac{3}{4} \times \frac{4}{3} = 1$)

f. $n = \frac{7}{8} \times \frac{4}{3}$ (Use Property of One)

2. Now compare lines a and f. From these lines you can see that

$$\frac{7}{8} \div \frac{3}{4} = n = \frac{7}{8} \times \frac{4}{3}, \quad \text{or}$$

$$\frac{7}{8} \div \frac{3}{4} = \frac{7}{8} \times \frac{4}{3}$$

What do you observe?

3. Now explain each line of this example.

a. $4\frac{2}{3} \div 10\frac{1}{2} = n$

b. $\frac{14}{3} \div \frac{21}{2} = n$ (Rename $4\frac{2}{3}$ and $10\frac{1}{2}$)

c. $n \times \frac{21}{2} = \frac{14}{3}$ (Rewrite division sentence as multiplication sentence)

d. $(n \times \frac{21}{2}) \times \frac{2}{21} = \frac{14}{3} \times \frac{2}{21}$ (Multiply $(n \times \frac{21}{2})$ and $\frac{14}{3}$ by $\frac{2}{21}$)

e. $n \times (\frac{21}{2} \times \frac{2}{21}) = \frac{14}{3} \times \frac{2}{21}$ (Use Associative Property)

f. $n \times 1 = \frac{14}{3} \times \frac{2}{21}$ ($\frac{21}{2} \times \frac{2}{21} = 1$)

g. $n = \frac{14}{3} \times \frac{2}{21}$ (Use Property of One)

h. $\frac{14}{3} \div \frac{21}{2} = \frac{14}{3} \times \frac{2}{21}$ (From b and g)

4. What relation does line h suggest?

5. Suppose you apply your observations in exercises 2 and 4 and write this sentence:

$$\frac{5}{8} \div \frac{3}{4} = \frac{5}{8} \times \frac{4}{3}$$

You can test the correctness of this sentence by using the meaning of division. That is,

Does $(\frac{5}{8} \times \frac{4}{3}) \times \frac{3}{4} = \frac{5}{8}$?

Does $\frac{20}{24} \times \frac{3}{4} = \frac{5}{8}$?

Does $\frac{60}{96} = \frac{5}{8}$?

$$\frac{60}{96} = \frac{2 \times 2 \times 3 \times 5}{2 \times 2 \times 2 \times 2 \times 2 \times 3} = \frac{5}{2 \times 2 \times 2} = \frac{5}{8}$$

So the quotient $\frac{5}{8} \div \frac{3}{4}$ is the same number as the product $\frac{5}{8} \times \frac{4}{3}$.

Do you see an easier way to show that $(\frac{5}{8} \times \frac{4}{3}) \times \frac{3}{4} = \frac{5}{8}$?
(Use Associative Property: $\frac{5}{8} \times (\frac{4}{3} \times \frac{3}{4}) = \frac{5}{8}$)
True: $\frac{5}{8} \times 1 = \frac{5}{8}$

6. Exercises 2, 4, and 5 suggest that

a. $\frac{3}{4} \div \frac{9}{10} = \frac{3}{4} \times \underline{(\frac{10}{9})}$.

b. $\frac{12}{5} \div \frac{7}{16} = \underline{(\frac{12}{5}} \times \underline{(\frac{16}{7})}$.

c. $\frac{1}{7} \div \frac{1}{10} = \underline{(\frac{1}{7})} \times \underline{(\frac{10}{1})}$.

d. $3\frac{5}{8} \div 2\frac{1}{4} = \frac{29}{8} \times \underline{(\frac{4}{9})}$.

e. $9 \div 6\frac{1}{5} = \underline{(\frac{9}{1})} \times \underline{(\frac{5}{31})}$.

f. $10\frac{3}{4} \div 12 = \underline{(\frac{43}{4})} \times \underline{(\frac{1}{12})}$.

7. Write the multiplication sentence which states the same relationship as each division sentence.

a. $\frac{17}{5} \div \frac{6}{7} = \frac{17}{5} \times \frac{7}{6} (\frac{17}{5} \times \frac{7}{6}) \times \frac{6}{7} = \frac{17}{5})$ c. $\frac{4}{21} \div \frac{3}{7} = \frac{21}{4} \times \frac{3}{7} ((\frac{21}{4} \times \frac{3}{7}) \times \frac{3}{7} = \frac{4}{21})$

b. $8\frac{3}{4} \div 2\frac{1}{2} = \frac{35}{4} \times \frac{2}{5} ((\frac{35}{4} \times \frac{2}{5}) \times 2\frac{1}{2} = 8\frac{3}{4})$ d. $5\frac{3}{8} \div 9\frac{1}{3} = \frac{43}{8} \times \frac{3}{28} ((\frac{43}{8} \times \frac{3}{28}) \times 9\frac{1}{3} = 5\frac{3}{8})$

8. Test each multiplication sentence you wrote for exercise 7 to see whether or not the sentence is true.

Was any sentence false? (c) If so, can you change any numbers to make a true sentence? $(\frac{21}{4} \div \frac{3}{7} = \frac{21}{4} \times \frac{7}{3})$

9. Write a statement describing a short way to divide by a rational number. (To divide a number by a rational number, multiply the number by the reciprocal of the divisor.)

10. If $b \neq 0$, $c \neq 0$, $d \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{?}{?}$ $(\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c})$

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}}$$

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Exercise Set 11

Use $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ to find simple names for these quotients.

1. $\frac{5}{8} \div \frac{3}{7}$ $\left(\frac{35}{24}\right)$
2. $3\frac{5}{8} \div 2\frac{1}{3}$ $\left(\frac{87}{56}\right)$
3. $7 \div \frac{3}{5}$ $\left(\frac{35}{3}\right)$
4. $\frac{24}{16} \div \frac{6}{4}$ $\left(\frac{96}{96}$ or $1\right)$
5. $\frac{9}{4} \div \frac{4}{9}$ $\left(\frac{81}{16}\right)$
6. $7\frac{1}{2} \div 12$ $\left(\frac{15}{24}$ or $\frac{5}{8}\right)$
7. $5280 \div 16\frac{1}{2}$ (320)
8. $320 \div 144$ $\left(2\frac{2}{9}\right)$
9. $16\frac{1}{2} \div 3$ $\left(\frac{11}{2}\right)$
10. $14 \div 2\frac{1}{2}$ $\left(\frac{28}{5}\right)$

11. Match each quotient expression in Column A with the product expression which names the same number in Column B.

Column A		Column B
a. $\frac{3}{4} \div \frac{4}{3}$	(f)	e. $\frac{4}{3} \times \frac{4}{3}$
b. $\frac{3}{4} \div \frac{3}{4}$	(g)	f. $\frac{3}{4} \times \frac{3}{4}$
c. $\frac{4}{3} \div \frac{4}{3}$	(h)	g. $\frac{3}{4} \times \frac{4}{3}$
d. $\frac{4}{3} \div \frac{3}{4}$	(e)	h. $\frac{4}{3} \times \frac{3}{4}$

12. Complete:

a. $6 \div \frac{3}{4} = 6 \times \underline{(\frac{4}{3})} = \frac{24}{3} = 8$

b. $\frac{5}{3} \div \frac{5}{6} = \frac{5}{3} \times \underline{(\frac{6}{5})} = \frac{30}{15} = \underline{(2)}$

c. $\frac{3}{4} \div \underline{(\frac{1}{8})} = \frac{3}{4} \times 8 = \frac{24}{4} = \underline{(6)}$

d. $\frac{1}{2} \div \frac{1}{2} = \frac{1}{2} \times \underline{(2)} = \underline{(1)}$

e. $\frac{12}{8} \div \frac{3}{2} = \frac{12}{8} \times \underline{(\frac{2}{3})} = \underline{(1)}$

Find a fraction or mixed form for n which makes each sentence true.

13. $9\frac{7}{8} \div 2\frac{3}{4} = n$ $(\frac{79}{22})$

14. $12\frac{1}{2} \div 50 = n$ $(\frac{1}{4})$

15. $10 \div 3\frac{1}{7} = n$ $(\frac{70}{22})$

16. $1 \div 2\frac{2}{10} = n$ $(\frac{10}{22})$

17. $8\frac{1}{2} \div \frac{3}{5} = n$ $(\frac{85}{6})$

18. $19\frac{1}{3} \div 3 = n$ $(\frac{58}{9})$

19. $3\frac{4}{15} \div 4\frac{2}{3} = n$ $(\frac{7}{10})$

20. $100 \div 2\frac{1}{4} = n$ $(\frac{400}{9})$

21. $2\frac{3}{10} \div 4 = n$ $(\frac{23}{40})$

22. $16\frac{4}{5} \div 4\frac{9}{10} = n$ $(\frac{24}{7})$

23. Write out work for exercise 13 as shown in exercise 1 of Exercise Set 10.

$(9\frac{7}{8} \div 2\frac{3}{4} = n$

$n \times 2\frac{3}{4} = 9\frac{7}{8}$

$(n \times \frac{11}{4}) \times \frac{4}{11} = \frac{79}{8} \times \frac{4}{11}$

$n \times (\frac{11}{4} \times \frac{4}{11}) = \frac{79}{8} \times \frac{4}{11}$

$n \times 1 = \frac{79}{8} \times \frac{4}{11}$

$n = \frac{316}{88}$ or $\frac{79}{22}$

$9\frac{7}{8} \div 2\frac{3}{4} = \frac{79}{22})$

24. Write out exercise 19 as shown in exercise 1 of Exercise Set 10.

$(3\frac{4}{5} \div 4\frac{2}{3} = n$

$n \times 4\frac{2}{3} = 3\frac{4}{15}$

$(n \times \frac{14}{3}) \times \frac{3}{14} = \frac{49}{15} \times \frac{3}{14}$

$n \times (\frac{14}{3} \times \frac{3}{14} = \frac{49}{15} \times \frac{3}{14})$

$n \times 1 = \frac{49}{15} \times \frac{3}{14}$

$n = \frac{147}{210}$ or $\frac{7}{10}$

$3\frac{4}{15} \div 4\frac{2}{3} = \frac{7}{10}$

III. PROBLEMS SOLVED BY DIVISION

DIVISION ON THE NUMBER LINE

Objectives: To help pupils recognize problem situations which can be solved by division of rational numbers.

To picture the operation of division of rational numbers.

To relate the computational process for finding quotients, i.e., multiplying by the reciprocal of the divisor, to a typical problem situation requiring division.

To show that division of rational numbers on the number line may be thought of as changing the unit segment on the number line.

Suggested Teaching Procedure:

The pupils now have a process for computing quotients. In the section "Problems Solved by Division," the Exploration presents a problem situation to be solved by using a diagram which is essentially a part of a number line. Division is pictured as a process of finding how many times the segment which pictures the divisor can be laid off on the segment representing the dividend. The numbers chosen for this discussion are such that for each quotient the divisor is less than the dividend, but in the next section larger divisors are used also.

Attention is centered on the segment with measure 1, to emphasize the significance of the reciprocal of the divisor in the problem situation. In example 2 of the Exploration, $1\frac{1}{2}$ hops or $\frac{3}{2}$ hops are required to cover 1 foot. For a mathematical sentence we have

$$\frac{3}{2} \times \frac{2}{3} = 1$$

or $$1\frac{1}{2} \text{ (hops)} \times \frac{2}{3} \text{ (foot)} = 1 \text{ (foot)}$$

Before beginning the Exploration for the section, "Problems Solved by Division," you may wish to conduct a discussion along these lines:

Imagine you have six oranges on your desk. Also on your desk are a stack of plates and a knife. Find how many people you can serve $\frac{3}{4}$ of an orange.

Imagine you pick up the knife and cut the oranges in such a way that you prepare the servings. Think about what you would do. In a few minutes I will ask several of you to describe the method you would follow.

> Give the children time to think about their plans of action.
>
> Two general procedures for finding that 8 servings may be made will probably be given. Each can be written in symbols to show division.

How many servings of $\frac{3}{4}$ orange could be made from 6 oranges?

> All should give answers with general agreement on 8 servings.

Describe the procedure you planned for cutting and preparing the servings.

> Listen to the procedures and summarize them in a way similar to the following: (Be sure all of the information given below is included. If it is not, question the children until it is all given.)
>
> (1) I cut each of the six oranges into four equal pieces. Then I had 24 pieces, each one-fourth of an orange. The servings were to be $\frac{3}{4}$ of an orange, so I put 3 pieces on each plate. $24 \div 3 = 8$. I made 8 servings.
>
> (2) I cut one orange into four equal pieces. Then I put $\frac{3}{4}$ of the orange on one plate and $\frac{1}{4}$ of an orange on another plate. Then I cut a second orange in the same way and put $\frac{3}{4}$ on one plate and $\frac{1}{4}$ of an orange on the plate with the other $\frac{1}{4}$ ... I then had 8 servings.

> Pupils may wish to use sketches on the board to supplement their explorations.

I will write in symbols the procedure some of you used, first cutting all the oranges into fourths. The problem is:

$$\frac{6}{1} \div \frac{3}{4}$$

The six oranges were changed from $\frac{6}{1}$ oranges to $\frac{24}{4}$ oranges by cutting each orange into $\frac{4}{4}$. I will restate the problem by renaming $\frac{6}{1}$ as $\frac{24}{4}$. Now the problem is:

$$\frac{24}{4} \div \frac{3}{4}$$

You then said that since there were 24 fourths, and 3 fourths were to be put on each plate, 24 should be divided by 3. The problem can be restated as,

$$24 \div 3 = 8$$

Summarized, the method you followed is:

$$\frac{6}{1} \div \frac{3}{4}$$

$$\frac{24}{4} \div \frac{3}{4}$$

$$24 \div 3 = 8$$

This is a different method for dividing rational numbers from the one you have studied. You first rename the numbers by fractions with a common denominator and divide the numerator of the product by the numerator of the known factor. Now I will write in symbols the procedure you described for cutting one orange at a time. The problem is:

$$\frac{6}{1} \div \frac{3}{4}$$

One orange was cut so that $\frac{3}{4}$ orange was put on one plate and $\frac{1}{4}$ orange on another plate. How many servings were made from one orange? (One whole serving and part of another.) What part of another serving? ($\frac{1}{3}$) Then how many servings were made from one orange? ($1\frac{1}{3}$)

> The teacher should emphasize that $\frac{1}{4}$ orange is $\frac{1}{3}$ of a three-fourths serving. As with the strips of paper suggested earlier there is a change in unit-- from one orange as unit to one serving as unit.

This means that $1 \div \frac{3}{4} = 1\frac{1}{3}$ or $\frac{4}{3}$ servings. How many servings will 6 oranges make? (They will make 6 times as many. $\frac{6}{1} \times \frac{4}{3} = \frac{24}{3} = 8$.) In symbols then, our problem becomes,

$$\frac{6}{1} \div \frac{3}{4} = \frac{6}{1} \times \frac{4}{3} = \frac{24}{3} = 8$$

What number is $1 \div \frac{3}{4}$? ($\frac{4}{3}$) Since you know how many servings there are in one orange, how did you find how many servings 6 oranges make? (There are 6 times as many in 6 as there are in 1.) Write this in symbols.

$$(6 \div \frac{3}{4} = 6 \times \frac{4}{3} = \frac{24}{3} = 8.)$$

Can you find $12 \div \frac{3}{4}$ in this way without cutting oranges? (Yes. We know that $\frac{4}{3} \times \frac{3}{4} = 1$, so $12 \times (\frac{4}{3} \times \frac{3}{4}) = 12$. Write this in symbols.)

$$(12 \div \frac{3}{4} = 12 \times \frac{4}{3} = \frac{48}{3} = 16.)$$

Is this really the same method you have learned for computing quotients of rational numbers? (Yes)

In the section, "Division on the Number Line," the operation of division is again examined as a matter of change of scale.

Points corresponding to the dividend and divisor are labeled and the segments whose measures are these numbers are identified. A new scale is then laid out, with the "divisor segment" as unit segment, and the number corresponding to the dividend in the new scale is determined. This number is the required quotient.

The sentences preceding the exercises in the Exploration show that the quotient of $m\overline{AC} \div m\overline{AB}$ is the same number regardless of the scale used. That is if $\overline{AC}$ is the union of segment $\overline{AB}$ and another segment half as long as $\overline{AB}$, the sentence

$$m\overline{AC} \div m\overline{AB} = \frac{3}{2}$$

is true if the computation is done with measures obtained from either of the two scales shown. Using the original scale, $m\overline{AC} = 1$ and $m\overline{AB} = \frac{2}{3}$, and

$$1 \div \frac{2}{3} = \frac{3}{2}$$

Using the new scale, $m\overline{AC} = \frac{3}{2}$ and $m\overline{AB} = 1$

Again $$\frac{3}{2} \div 1 = \frac{3}{2}.$$

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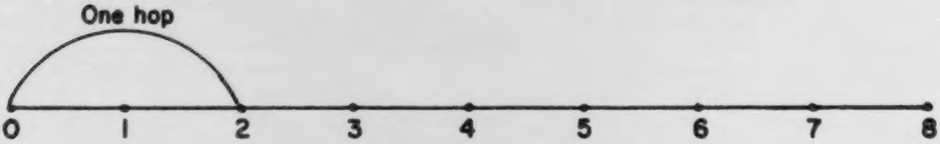
PROBLEMS SOLVED BY DIVISION

Exploration

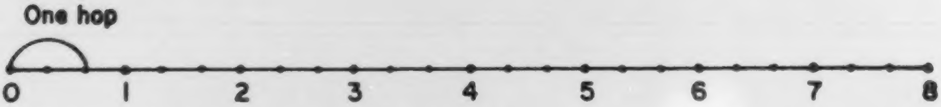
You have learned how to divide by a rational number. Now think about what this means when you have a problem to solve in which the numbers are rational numbers.

1. A mechanical toy moves by hops. In each hop it covers 2 feet. How many hops will it take to cover 8 feet?



You know it will take 4 hops. How would you show this on the diagram? (Draw arcs from 2 to 4, 4 to 6, 6 to 8) What mathematical operation on the number 8 and 2 gives you the answer 4? (Division) The mathematical sentence is $8 \div 2 = n$ or $2 \times n = 8$.

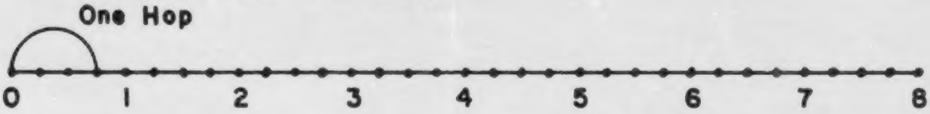
2. Now suppose you have a small copy of the toy. This one covers $\frac{2}{3}$ foot in each hop. How many hops does it make to cover 8 feet?



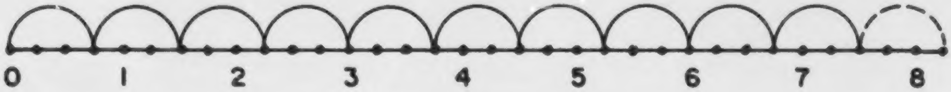
Count on the diagram to find the answer. (Notice each unit segment is separated into three congruent segments.) Did you get 12?

Is this the same kind of problem as exercise 1? (yes) What is the mathematical sentence for it? ($n \times \frac{2}{3} = 8$ or $8 \div \frac{2}{3} = n$)

3. Another of these toys hops $\frac{3}{4}$ foot. How many hops will it take to cover 8 feet?



Count on the diagram to find the answer. (Notice each unit segment is separated into four congruent segments.) Is the answer a whole number? (no)



Ten hops take the toy $7\frac{1}{2}$ feet. The next hop will take it beyond 8 feet. What "part of a hop" will take it just to 8 feet? ($\frac{2}{3}$ hop)

What is the mathematical sentence for this problem?
($8 \div \frac{3}{4} = n$ or $n \times \frac{3}{4} = 8$)

Now look at the mathematical sentences and the solutions you have found from the diagrams.

1. $8 \div 2 = n$ $\quad$ $n = 4$

2. $8 \div \frac{2}{3} = n$ $\quad$ $n = 12$

3. $8 \div \frac{3}{4} = n$ $\quad$ $n = 10\frac{2}{3}$

4. Find names for these quotients by using $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

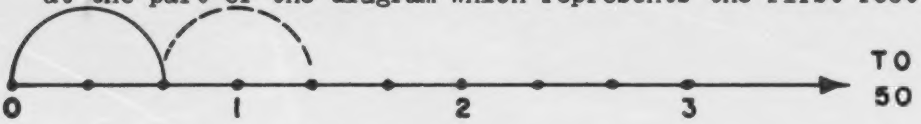
a. $8 \div 2$
($8 \div 2 = \frac{8}{1} \times \frac{1}{2} = \frac{8}{2} = 4$)

b. $8 \div \frac{2}{3}$
($8 \div \frac{2}{3} = \frac{8}{1} \times \frac{3}{2} = \frac{24}{2} = 12$)

c. $8 \div \frac{3}{4}$
($8 \div \frac{3}{4} = \frac{8}{1} \times \frac{4}{3} = \frac{32}{3} = 10\frac{2}{3}$)

Do your results agree with your answers for exercises 1-3? (yes)

5. Look again at exercise 2. If this toy had to go 50 feet, it would be awkward to find the answer from a diagram. Look at the part of the diagram which represents the first foot.



The toy covers $\frac{2}{3}$ foot in its first hop. The second hop will take it past 1. What "part of a hop" will take it to 1? $(\frac{1}{2})$ To cover 1 foot, the toy takes $1\frac{1}{2}$ hops, or $\frac{3}{2}$ hops. If you know this, how can you find how many hops it will take for 2 feet? $(2 \times \frac{3}{2})$ for 5 feet? $(5 \times \frac{3}{2})$ for 50 feet? $(50 \times \frac{3}{2})$ Check the following on the diagram:

1 foot	$\frac{3}{2}$ hops
2 feet	$2 \times \frac{3}{2}$
3 feet	$3 \times \frac{3}{2}$

6. Write the mathematical sentence for this problem. If a toy hops $\frac{2}{3}$ feet in one hop, how many hops must it make to go 50 feet? $(50 \div \frac{2}{3} = n)$

7. Solve the sentence $50 + \frac{2}{3} = n$. Use $\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.
$(50 \div 23 = 50 \times \frac{3}{2}$
$= \frac{150}{2} = 75)$

8. $50 + \frac{2}{3} = \frac{50}{1} \times \frac{3}{2}$. In the diagram in exercise 5, what does $\frac{3}{2}$ represent? (The number of hops to go 1 foot)

9. Look again at exercise 3. The sentence for this problem was

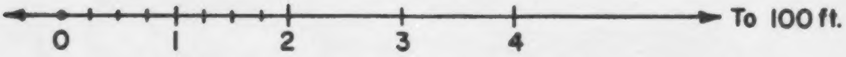
$$8 \div \frac{3}{4} = n$$

$$8 \div \frac{3}{4} = \frac{8}{1} \times \frac{4}{3}.$$

What does the number $\frac{4}{3}$ represent in the diagram?

(The number of hops to go 1 foot)

10. a. Trace the number line below and use it to make the first part of a diagram to represent the hops of a toy which hops $1\frac{3}{4}$ feet per hop and has 100 feet to go.



b. Write the mathematical sentence showing the relationship of the 100 feet, the $1\frac{3}{4}$ feet, and the number of hops. $(100 \div 1\frac{3}{4} = n)$

c. Solve the mathematical sentence using $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

$(100 \div 1\frac{3}{4} = \frac{100}{1} \times \frac{4}{7} = \frac{400}{7} = 57\frac{1}{7})$

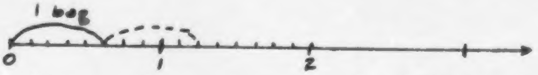
What does $\frac{d}{c}$ represent in the diagram?

(Number of hops needed to go 1 foot)

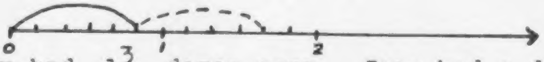
Exercise Set 12

For each problem, make part or all of a line diagram. Then write the mathematical sentence. Solve the sentence and answer the question.

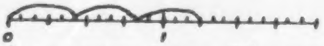
1. A store had 75 pounds of candy. The owner decided to put it in bags containing $\frac{5}{8}$ pounds each. How many bags could he fill? ($75 \div \frac{5}{8} = n$, or $n \times \frac{5}{8} = 75$: He could fill 120 bags)



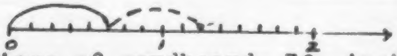
2. Tom watched a bug crawl up a wall 9 feet high. It climbed $\frac{5}{6}$ foot in a minute. At that rate, how long would it take the bug to reach the top? ($9 \div \frac{5}{6} = n$; or $n \times \frac{5}{6} = 9$: It would take $10\frac{4}{5}$ min.)



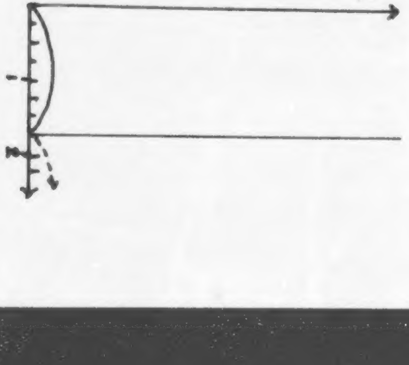
3. Joan's mother had $1\frac{3}{4}$ dozen eggs. Joan had a dessert recipe which called for $\frac{5}{12}$ dozen. She wanted to make as much dessert as possible for a party. How many recipes could she make if she used all the eggs? ($1\frac{3}{4} \div \frac{5}{12} = n$; or $n \times \frac{5}{12} = 1\frac{3}{4}$: She could make $4\frac{1}{5}$ recipes)



4. A frozen food company has 50 pounds of peas. How many $\frac{5}{8}$ pound packages can they make? ($50 \div \frac{5}{8} = n$; or $n \times \frac{5}{8} = 50$: They could make 80 packages)



5. You have a piece of cardboard 30 inches long and 10 inches wide. How many 10-inch strips can you cut from the piece if you make the strips $1\frac{3}{4}$ inches wide? ($30 \div 1\frac{3}{4} = n$ or $n \times 1\frac{3}{4} = 30$. You could cut 17 strips)



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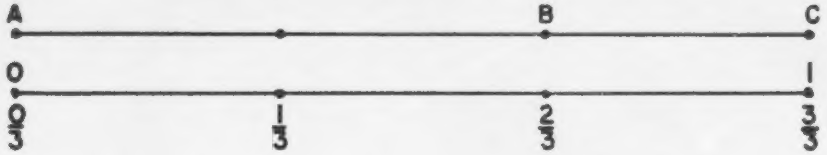
DIVISION ON THE NUMBER LINE

Exploration

You can picture division on the number line if you think of two different scales on the same line. Consider $1 \div \frac{2}{3}$.

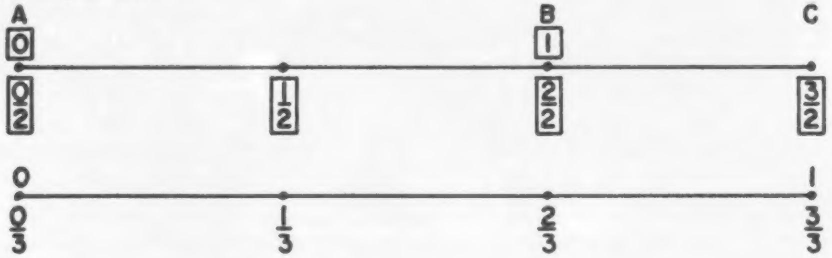


In the diagram, what segment has measure 1? $(\overline{AC})$

What segment has measure $\frac{2}{3}$? $(\overline{AB})$

Now think about a second scale on the number line in which $\overline{AB}$ has measure 1.

On the diagram below labels for the new scale are written on $\overline{AC}$ in squares.



How many times can the new unit segment $(\overline{AB})$ be laid off on $\overline{AC}$? $(1\frac{1}{2})$

The picture shows that $m\overline{AC} \div m\overline{AB} = \frac{3}{2}$

In the original scale, $1 \div \frac{2}{3} = \frac{3}{2}$

In the new scale, $\frac{3}{2} \div 1 = \frac{3}{2}$

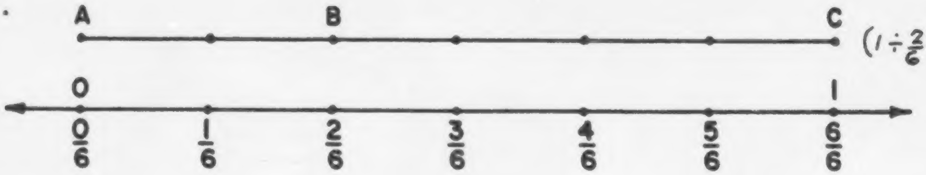
The picture also shows that $m\overline{AB} + m\overline{AC} = \frac{2}{3}$

In the original scale, $\frac{2}{3} + 1 = \frac{2}{3}$

In the new scale, $1 + \frac{3}{2} = \frac{2}{3}$

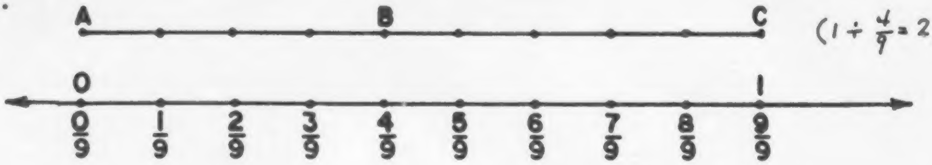
What division sentence is pictured on each number line below? In each, consider $m\overline{AC}$ as a product and $m\overline{AB}$ as the known factor.

1.


A
B
C
0
1
0/6
1/6
2/6
3/6
4/6
5/6
6/6

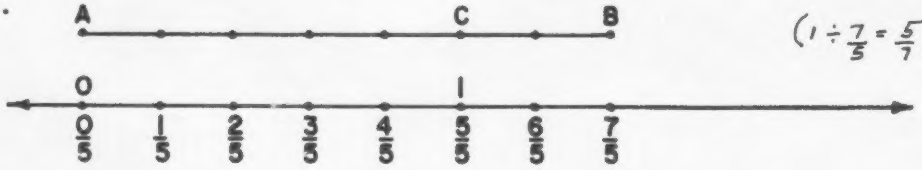
$(1 \div \frac{2}{6} = 3)$

2.


A
B
C
0
1
0/9
1/9
2/9
3/9
4/9
5/9
6/9
7/9
8/9
9/9

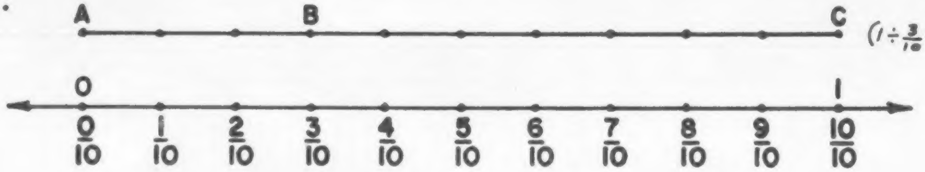
$(1 \div \frac{4}{9} = 2\frac{1}{4})$

3.


A
C
B
0
1
0/5
1/5
2/5
3/5
4/5
5/5
6/5
7/5

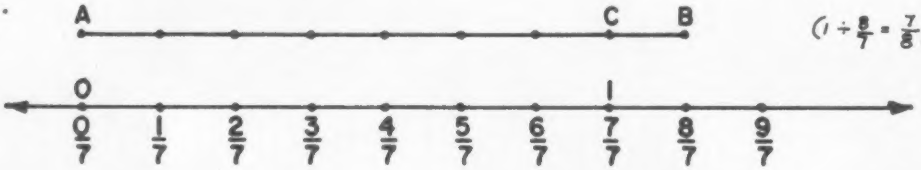
$(1 \div \frac{7}{5} = \frac{5}{7})$

4.


A
B
C
0
1
0/10
1/10
2/10
3/10
4/10
5/10
6/10
7/10
8/10
9/10
10/10

$(1 \div \frac{3}{10} = 3\frac{1}{3})$

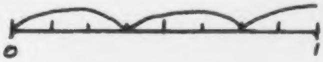
5.


A
C
B
0
1
0/7
1/7
2/7
3/7
4/7
5/7
6/7
7/7
8/7
9/7

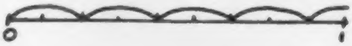
$(1 \div \frac{8}{7} = \frac{7}{8})$

Write a division sentence for each problem. Picture each one on a number line.

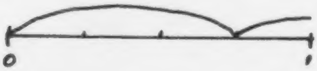
5. You have a measuring cup full of orange juice. You serve it in small glasses, each holding $\frac{3}{8}$ cup. How many servings are there in 1 cup? $(1 \div \frac{3}{8} = 2\frac{2}{3})$



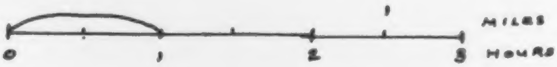
6. You have 1 yard of ribbon which you cut to make badges, $\frac{2}{9}$ of a yard for each badge. You have enough for how many badges? $(1 \div \frac{2}{9} = 4\frac{1}{2})$



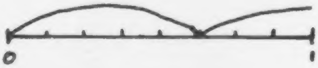
7. You have one cup of milk to make candy. Your recipe calls for $\frac{3}{4}$ cup. How many recipes can you make if you use all of the milk? $(1 \div \frac{3}{4} = 1\frac{1}{3})$



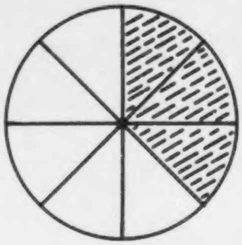
8. You have 1 mile to walk. You can walk $\frac{5}{2}$ mile in an hour. How long will it take you to walk 1 mile? $(1 \div \frac{5}{2} = \frac{2}{5})$



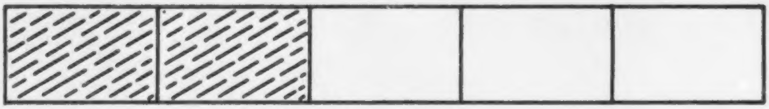
9. In many countries distances are measured in kilometers, rather than in miles. One kilometer is about $\frac{5}{8}$ mile. If you walk 1 mile, about how many kilometers have you walked? $(1 \div \frac{5}{8} = 1\frac{3}{5})$



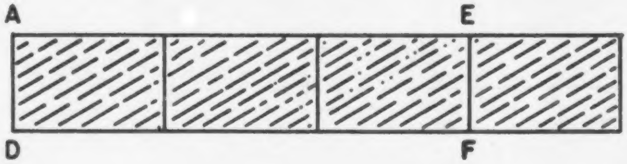
10. How many shaded regions are needed to cover the unit circular region? $(2\frac{2}{3};\ 1 \div \frac{3}{8} = 2\frac{2}{3})$



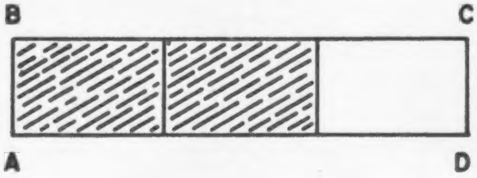
11. How many shaded regions will it take to cover the unit rectangular region? $(2\frac{1}{2};\ 1 \div \frac{2}{5} = 2\frac{1}{2})$



12. How many shaded regions are needed to cover region AEFD? $(\frac{3}{4};\ \frac{3}{4} \div 1 = \frac{3}{4})$



13. If the measure of region ABCD is 1, how many shaded regions will cover a region whose measure is 6? $(9;\ 6 \div \frac{2}{3} = 9)$



Exercise Set 13

1. You have a 90 mile trip to make. How many miles must you travel per hour, on the average, to finish in

   a. 3 hours? (30) What operation did you use? (Division)

   b. 2 hours? (45)

   c. $2\frac{1}{2}$ hours? (36) What operation? (Division)

   d. $1\frac{3}{4}$ hours? ($51\frac{3}{7}$)

   e. Suppose you are a racing driver. If you drive a 90-mile course in $\frac{9}{10}$ of an hour, what is your average number of miles per hour? (100 m.p.h.)

2. Express each measurement in the unit named.

   a. $34\frac{1}{2}$ ounces is ($2\frac{5}{32}$) pounds.

   b. $8\frac{1}{2}$ inches is ($\frac{17}{24}$) feet.

   c. $9\frac{7}{10}$ feet is ($3\frac{7}{30}$) yards.

   d. $9\frac{1}{4}$ pints is ($4\frac{5}{8}$) quarts.

   e. $24\frac{3}{5}$ seconds is ($\frac{41}{100}$) minutes.

3. You know that the measure of a rectangular region is the product of the measures of two adjacent sides.

Find in the table below the missing measure for each region.

Rectangular Region	Measure of First Side	Measure of Second Side	Measure of Region
A	$2\frac{5}{8}$	$(1\frac{13}{42})$	$3\frac{7}{16}$
B	$15\frac{3}{4}$	$12\frac{1}{2}$	$196\frac{7}{8}$
C	$(\frac{9}{14})$	$\frac{7}{12}$	$\frac{3}{8}$
D	$\frac{3}{4}$	$(2\frac{2}{5})$	$1\frac{4}{5}$
E	$(1\frac{21}{31})$	$3\frac{7}{8}$	$6\frac{1}{2}$

Write a mathematical sentence for each problem. Be sure to answer the question in a complete sentence.

4. A peanut vendor at the ball park bagged 500 pounds of peanuts, putting about $\frac{3}{8}$ pound in each bag. How many bags could he fill from 1 pound? $(2\frac{2}{3})$ From the 500 pounds? $(1333\frac{1}{3})$

5. One of the satellites takes $1\frac{3}{5}$ hours to travel around the earth. How many trips does it make in a day? (15)

6. The Sault Sainte Marie Canal is 1.2 miles long and the Welland Canal is 27.6 miles long. The Welland Canal is how many times as long as the Sault Sainte Marie Canal? (Use fraction names to compute.)

$\left(\frac{27.6}{1.2} = 23\right)$

7. Plans are made to use jacks to raise the temple in Egypt called Abu Simbel 200 feet, in stages of .04 inches. At this rate, how many stages will be required? (5000)

8. A club bought 50 yards of material to make towels for a bazaar. If each towel requires $\frac{7}{8}$ yard of material, how many towels can be made? $\left(57\frac{1}{7}\right)$

9. A factory workman can complete one article in $2\frac{1}{2}$ minutes. How many articles can he complete in 8 hours if there is no loss of time? (192)

10. If a girl can knit $1\frac{3}{4}$ inches of a scarf in one hour, how many hours will it take to complete a 35 inch scarf? (20 hrs.)

Exercise Set 14

1. Name each number by a decimal.

a. $\frac{17}{100}$ (0.17) c. $\frac{6}{20}$ (0.3) e. $\frac{49}{50}$ (0.98) g. $1\frac{11}{10}$ (2.1)

b. $\frac{1}{4}$ (0.25) d. $\frac{1}{2}$ (0.5) f. $\frac{75}{250}$ (0.3) h. $3\frac{6}{5}$ (4.2)

2. Find the simplest fraction name for each number.

a. 0.12 ($\frac{3}{25}$) c. 1.010 ($1\frac{1}{100}$) e. 3.19 ($3\frac{19}{100}$) g. 9.9 ($9\frac{9}{10}$)

b. 0.07 ($\frac{7}{100}$) d. 0.101 ($\frac{101}{1000}$) f. 0.006 ($\frac{3}{500}$) h. 4.02 ($4\frac{1}{50}$)

3. Write in the simplest fraction or mixed form.

a. $\frac{90}{10}$ (9) c. $\frac{430}{100}$ ($4\frac{3}{10}$) e. $\frac{8}{24}$ ($\frac{1}{3}$) g. $\frac{6}{18}$ ($\frac{1}{3}$)

b. $2\frac{7}{6}$ ($3\frac{1}{6}$) d. $\frac{12}{5}$ ($2\frac{2}{5}$) f. $\frac{609}{100}$ ($6\frac{9}{100}$) h. $\frac{2000}{1000}$ (2)

4. Name each of these numbers by a fraction. Use the simplest fraction name.

a. $5\frac{1}{2}$ ($\frac{11}{2}$) c. 0.20 ($\frac{1}{5}$) e. $1\frac{5}{4}$ ($\frac{9}{4}$) g. $7\frac{3}{8}$ ($\frac{59}{8}$)

b. $6\frac{6}{6}$ ($\frac{7}{1}$) d. 0.750 ($\frac{3}{4}$) f. 4.1 ($\frac{41}{10}$) h. 12.01 ($\frac{1201}{100}$)

5. Arrange in order from greatest to least.

1.1; $\frac{1}{10}$; 2.11; 1.01; $\frac{3}{1000}$; 0.101; 0.11

(2.11, 1.1, 1.01, 0.11, 0.101, $\frac{1}{10}$, $\frac{3}{1000}$.)

6. Write the reciprocal of each number.

a. $\frac{1}{4}$ $(\frac{4}{1})$ c. $\frac{7}{10}$ $(\frac{10}{7})$ e. $\frac{1}{1}$ $(\frac{1}{1})$ g. $\frac{13}{9}$ $(\frac{9}{13})$

b. 2 $(\frac{1}{2})$ d. $1\frac{5}{9}$ $(\frac{9}{14})$ f. $3\frac{2}{7}$ $(\frac{7}{23})$ h. $15\frac{4}{7}$ $(\frac{7}{109})$

7. Add each pair of numbers and express the result in simplest form.

a. $\begin{array}{c}\frac{2}{3}\\ \frac{1}{2}\\ \hline (1\frac{1}{6})\end{array}$ c. $\begin{array}{c}\frac{3}{4}\\ \frac{2}{3}\\ \hline (1\frac{5}{12})\end{array}$ e. $\begin{array}{c}1\frac{5}{6}\\ 2\frac{3}{8}\\ \hline (4\frac{5}{24})\end{array}$ g. $\begin{array}{c}7\frac{1}{6}\\ 4\frac{5}{8}\\ \hline (11\frac{19}{24})\end{array}$

b. $\begin{array}{c}12\frac{3}{4}\\ 1\frac{1}{3}\\ \hline (14\frac{1}{12})\end{array}$ d. $\begin{array}{c}4\frac{5}{6}\\ \frac{1}{4}\\ \hline (5\frac{1}{12})\end{array}$ f. $\begin{array}{c}8\frac{1}{8}\\ 2\frac{7}{10}\\ \hline (10\frac{33}{40})\end{array}$ h. $\begin{array}{c}2\frac{6}{7}\\ 7\frac{1}{2}\\ \hline (10\frac{5}{14})\end{array}$

8. Subtract each pair of numbers and express the result in the simplest fraction or mixed form.

a. $\begin{array}{c}\frac{12}{10}\\ 1\frac{1}{5}\\ \hline (0)\end{array}$ c. $\begin{array}{c}1\frac{1}{3}\\ 1\frac{1}{8}\\ \hline (\frac{5}{24})\end{array}$ e. $\begin{array}{c}4\frac{3}{3}\\ 3\frac{5}{5}\\ \hline (1)\end{array}$ g. $\begin{array}{c}14\frac{1}{9}\\ 5\frac{1}{2}\\ \hline (8\frac{11}{18})\end{array}$

b. $\begin{array}{c}1\frac{7}{10}\\ 1\frac{5}{12}\\ \hline (\frac{17}{60})\end{array}$ d. $\begin{array}{c}23\frac{3}{6}\\ 14\frac{3}{5}\\ \hline (8\frac{9}{10})\end{array}$ f. $\begin{array}{c}1\frac{1}{2}\\ \frac{3}{4}\\ \hline (\frac{3}{4})\end{array}$ h. $\begin{array}{c}6\frac{2}{3}\\ 5\frac{8}{7}\\ \hline (\frac{11}{21})\end{array}$

9. Find a simplest fraction name or mixed for for each product expression.

a. $2\frac{1}{2} \times \frac{5}{8}$ $(1\frac{9}{16})$ d. $\frac{5}{6} \times 8\frac{1}{4}$ $(6\frac{7}{8})$ g. $\frac{1}{9} \times \frac{1}{9}$ $(\frac{1}{81})$

b. $3\frac{1}{4} \times 4\frac{1}{3}$ $(14\frac{1}{12})$ e. $2\frac{2}{5} \times 2\frac{2}{7}$ $(5\frac{17}{35})$ h. $5\frac{3}{7} \times 1\frac{5}{9}$ $(8\frac{4}{9})$

c. $9\frac{7}{8} \times \frac{1}{2}$ $(4\frac{15}{16})$ f. $\frac{12}{2} \times \frac{10}{9}$ $(6\frac{2}{3})$ i. $\frac{19}{3} \times \frac{21}{1}$ (153)

10. Express each quotient in simplest form.

a. $7 \div \frac{3}{4}$ $(9\frac{1}{3})$ d. $9\frac{2}{3} \div 2\frac{3}{4}$ $(3\frac{17}{33})$ g. $5\frac{1}{2} \div 2$ $(2\frac{3}{4})$

b. $\frac{3}{4} \div \frac{2}{3}$ $(1\frac{1}{8})$ e. $14 \div 2\frac{2}{3}$ $(5\frac{1}{4})$ h. $21 \div \frac{2}{3}$ $(31\frac{1}{2})$

c. $\frac{7}{8} \div 4$ $(\frac{7}{32})$ f. $\frac{11}{12} \div 1\frac{1}{2}$ $(\frac{11}{18})$ i. $7\frac{5}{6} \div 3\frac{2}{3}$ $(2\frac{3}{22})$

j. Write out your work as shown in exercise 1 of Exercise Set 10:

$\frac{11}{5} \div \frac{7}{8}$ (See below)

J. $\frac{11}{5} \div \frac{7}{8} = n$

$n \times \frac{7}{8} = \frac{11}{5}$

$(n \times \frac{7}{8}) \times \frac{8}{7} = \frac{11}{5} \times \frac{8}{7}$

$n \times (\frac{7}{8} \times \frac{8}{7}) = \frac{11}{5} \times \frac{8}{7}$

$n \times 1 = \frac{11}{5} \times \frac{8}{7}$

$n = \frac{88}{35}$

$\frac{11}{5} \div \frac{7}{8} = \frac{88}{35}$

11. Complete with " >," "<," or " = ":

a. $\frac{7}{8} \times \frac{3}{4}$ __<__ $\frac{1}{4} + \frac{7}{12}$

b. $5\frac{1}{2} - 4\frac{11}{12}$ __>__ $2\frac{3}{4} \div 11$

c. $7\frac{1}{3} \div 2\frac{1}{5}$ __=__ $5 - 1\frac{2}{3}$

d. $2\frac{1}{2} + 3\frac{3}{8}$ __>__ $3\frac{3}{8} - 2\frac{5}{16}$

e. 8.4×0.74 __<__ 0.75×8.4

f. $\frac{5}{6} \div \frac{5}{12}$ __=__ $17.8 \div 8.9$

IV. RATIONAL NUMBERS AS QUOTIENTS OF WHOLE NUMBERS

EXTENDING THE SYSTEM OF FRACTIONS

Objectives: To show that the fraction symbol may be used to denote the quotient of two numbers.

To name the quotient of two whole numbers by a mixed form.

To extend the meaning of the fraction symbol to include fractions of which the numerator and denominator are rational numbers which are not whole numbers.

To find for a rational number named by such a fraction a fraction name with a whole-number numerator and denominator.

Suggested Teaching Procedure:

Recall that the fraction symbol may be defined as $\frac{\text{numeral}}{\text{numeral}}$. Heretofore we have used only fractions in which the numeral above the bar named a whole number and the numeral below the bar named a counting number. We have said that the meaning of the rational number named by the fraction can be pictured in relation to a unit segment, region, or set. Also, for any fraction, if we choose two points on a line to correspond to the numbers 0 and 1, we can find exactly one point on that number line which corresponds to the rational number named by the fraction. We now extend the meaning of the fraction symbol in two ways.

(1) In the section, "Rational Numbers as quotients of Whole Numbers", we show that it is consistent with what we know about rational numbers, and about the operation of division, to regard the fraction as a symbol for division. Thus $\frac{a}{b}$, where a names a whole number and b names a counting number, means $a \div b$. The relation of this idea to the "mixed form" is also discussed. Since the set of whole numbers is a subset of the set of rational numbers, we are now able to state the

result of the division of two whole numbers, such as $1267 \div 240$, as $5\frac{67}{240}$, as well as in the form used earlier, $1267 = (5 \times 240) + 67$. This idea is important for the discussion of computing quotients using decimal numerals.

It also reveals a property of rational numbers which whole numbers do not have. We have observed in study of division that the set of whole numbers under the operation of division does not have the Closure Property; i.e., the quotient of two whole numbers is not always a whole number. The set of rational numbers does have this property, except that division by zero is not possible.

(2) In the section, "Extending the System of fractions," the restriction that the two numerals in a fraction must be, respectively, a whole number and a counting number, is removed. We now permit the numerator to be any rational number, and the denominator to be any rational number except zero. To show that this extension does not seem to result in loss of the familiar properties, the renaming property ($\frac{a}{b} = \frac{a \times m}{b \times m}$, $b \neq 0$ and $m \neq 0$) and the multiplication property $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$, $b \neq 0$, $d \neq 0$) are shown to be true in examples using fractions with rational numbers as numerator and denominator. That these "new" fractions name rational numbers follows from the use of the fraction symbol to indicate division, and the fact that the rational numbers have the Closure Property under division (except division by 0).

When pupils have learned to work with fractions with numerator and denominator rational numbers, they frequently prefer to perform the operation of division as follows:

$$\frac{2}{3} \div \frac{4}{5} = \frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3} \times \frac{5}{4}}{\frac{4}{5} \times \frac{5}{4}} = \frac{\frac{2}{3} \times \frac{5}{4}}{1} = \frac{2}{3} \times \frac{5}{4} =$$

$$\frac{10}{12} = \frac{5}{6}$$

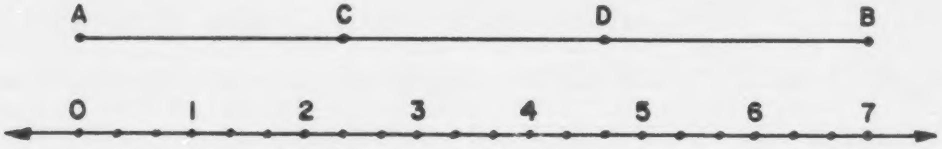
RATIONAL NUMBERS AS QUOTIENTS OF WHOLE NUMBERS

Exploration

The idea of a rational number and the operation of division of counting numbers are closely related.

Consider this question:

If a 7-inch segment is separated into 3 congruent segments, how long is each segment?



Since each segment is $\frac{1}{3}$ of $\overline{AB}$, its length in inches is

$$\frac{1}{3} \text{ of } 7, \text{ or } \frac{1}{3} \times \frac{7}{1}, \text{ or } \frac{7}{3}.$$

The language of the problem suggests that we can think of it in another way, as $7 \div 3$. Since whole numbers are also rational numbers,

$$\begin{aligned} 7 \div 3 &= \frac{7}{1} \div \frac{3}{1} \\ &= \frac{7}{1} \times \frac{1}{3} = \frac{7}{3}. \end{aligned}$$

Thus, one meaning for $\frac{a}{b}$, when a and b are whole numbers ($b \neq 0$), is $a \div b$.

Test this with $\frac{8}{5}$.

$$\begin{aligned} \frac{8}{5} = \frac{8 \times 1}{1 \times 5} = \frac{8}{1} \times \frac{1}{5} = \frac{8}{1} \div \frac{5}{1} \quad &\left(\text{Since } \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}\right) \\ = 8 \div 5. \end{aligned}$$

1. Express these quotients of counting numbers as rational numbers. Show your work as in exercise a.

   a. $18 \div 6 = \frac{18}{1} \times \frac{1}{6} = \frac{18}{6}$

   b. $28 \div 9 = \left(\frac{28}{1} \times \frac{1}{9} = \frac{28}{9}\right)$

   c. $5 \div 247 = \left(\frac{5}{1} \times \frac{1}{247} = \frac{6}{247}\right)$

   d. $46 \div 93 = \left(\frac{46}{1} \times \frac{1}{93} = \frac{46}{93}\right)$

   e. $1729 \div 1000$ $\left(1729 \times \frac{1}{1000} = \frac{1729}{1000}\right)$

   f. $a \div b$ $\left(a \times \frac{1}{b} = \frac{a}{b}\right)$

2. Express these rational numbers as quotients of counting numbers. Show your work.

   a. $\frac{4}{7} = \frac{4}{1} \times \frac{1}{7} = \frac{4}{1} \div \frac{7}{1} = 4 \div 7$

   b. $\frac{6}{13}$ $\left(= \frac{6}{1} \times \frac{1}{13} = \frac{6}{1} \div \frac{13}{1} = 6 \div 13\right)$

   c. $\frac{13}{8}$ $\left(= \frac{13}{1} \times \frac{1}{8} = \frac{16}{1} \div \frac{8}{1} = 13 \div 8\right)$

   d. $\frac{65}{127}$ $\left(= \frac{65}{1} \times \frac{1}{127} = \frac{65}{1} \div \frac{127}{1} = 65 \div 127\right)$

   e. $\frac{79}{12}$ $\left(\frac{79}{1} \times \frac{1}{12} = \frac{79}{1} \div \frac{12}{1} = 79 \div 12\right)$

   f. $\frac{m}{n}$ $\left(\frac{m}{1} \times \frac{1}{n} = \frac{m}{1} \div \frac{n}{1} = m \div n\right)$

3. Complete the following sentences.

   a. If a 2 hour period is divided into 5 periods of equal length, the length of each period is $\left(\frac{2}{5}\right)$ hours.

   b. A recipe calls for 3 cups of milk. To make half the recipe, $\left(\frac{3}{2}\right)$ cups of milk should be used.

   c. A man divided his garden into 5 parts of equal area. Each piece had area $\frac{576}{5}$ square feet. The area of the garden was (576) square feet.

   d. A rectangular field is 34 yards wide and has area 1700 square yards. The length of the field is $\left(\frac{1700}{34}\right.$ or $\left.50\right)$ yards.

   e. The quotient of two counting numbers is always a (rational) number.

By extending the operation of division to rational numbers we have also learned a new way to express the division process for whole numbers. In the fourth and fifth grades, you learned that to divide 31 by 3 meant to find n and r in the sentence

$$31 = (3 \times n) + r$$

so that n and r are whole numbers and $r < 3$.

If it turned out that $r = 0$ as in

$$33 = (3 \times n) + r$$

then we were actually finding an unknown factor.

Now we know that there is always a rational number missing factor. In other words there is a rational number p so that

$$31 \div 3 = p.$$

How is p related to n and r?

$$31 = (3 \times \overset{n}{10}) + \overset{r}{1}$$

$$31 \div 3 = \frac{31}{3} = \frac{(3 \times 10) + 1}{3} = \frac{3 \times 10}{3} + \frac{1}{3} = 10 + \frac{1}{3}$$

So $$p = \underset{n}{10}\overset{r}{\frac{1}{3}}$$

Here is another example: $146 \div 27$

$$\begin{array}{r} 5 \\ 27\,\big)\,\overline{146} \\ \underline{135} \\ 11 \end{array} \qquad\qquad 146 = (27 \times \underset{\overset{\uparrow}{n}}{5}) + \underset{\overset{\uparrow}{r}}{11}$$

Now we can express the division process using rational numbers.

$$\frac{146}{27} = \frac{(27 \times 5) + 11}{27} = \frac{27 \times 5}{27} + \frac{11}{27} = 5 + \frac{11}{27}$$

$$146 \div 27 = \underset{\overset{\uparrow}{n}}{5}\overset{\swarrow r}{\frac{11}{27}}$$

We also can express a simplest mixed form as a quotient of whole numbers.

$$6\frac{7}{18} = 6 + \frac{7}{18} = \frac{18 \times 6}{18} + \frac{7}{18} = \frac{(18 \times 6) + 7}{18} = \frac{115}{18}$$

So, $6\frac{7}{18} = 115 \div 18.$

Another way to express this relation is the sentence

$$115 = (18 \times 6) + 7$$

Exercise Set 15

1. Use the relation between division and rational numbers to show why each sentence is true.

Example: $(8 \times 39) \div 13 = 8 \times 3$

This is true because

$$(8 \times 39) \div 13 = \frac{8 \times 39}{13} = \frac{8 \times 3 \times 13}{13} = 8 \times 3$$

a. $(5 \times 12) \div 3 = 5 \times 4$ $(5 \times 4 \times 3) \div 3 = \frac{5 \times 4 \times 3}{3} = 5 \times 4$

b. $(7 \times 15) \div 5 = 7 \times 3$ $(7 \times 3 \times 5) \div 5 = \frac{7 \times 3 \times 5}{5} = 7 \times 3$

c. $(17 \times 2) \div 6 = 17 \div 3$ $(17 \times 2) \div (3 \times 2) = \frac{17 \times 2}{3 \times 2} = 17 \div 3$

d. $(3 \times 4) \div 20 = 3 \div 5$ $(3 \times 4) \div (5 \times 4) = \frac{3 \times 4}{5 \times 4} = 3 \div 5$

e. $(7^5 \div 7^2) = 7^3$ $(7 \times 7 \times 7 \times 7 \times 7) \div (7 \times 7) = 7 \times 7 \times 7$ or 7^3

f. $(5 \times 6^3) \div 6^2 = 5 \times 6$ $(5 \times 6 \times 6 \times 6) \div (6 \times 6) = 5 \times 6$

g. $(5 \times 6^2) \div 6^3 = 5 \div 6$ $(5 \times 6 \times 6) \div (6 \times 6 \times 6) = 5 \div 6$

h. $(11 \times \frac{2}{3}) = (11 \div 3) \times 2$ $\left(\frac{11 \times 2}{3}\right) = \frac{11}{3} \times \frac{2}{1} = (11 \div 3) \times 2$

i. $(11 \times \frac{2}{3}) = (11 \times 2) \div 3$ $\left(\frac{11 \times 2}{3}\right) = (11 \times 2) \div 3$

j. $(n \times .7) = (n \times 7) \div 10$ $\left(\frac{n \times 7}{10}\right) = \frac{n}{1} \times \frac{7}{10} = (n \times 7) \div 1$

k. $(n \times .76) = (n \times 76) \div 100$ $\left(\frac{n \times 76}{100}\right) = \frac{n}{1} \times \frac{76}{100} = (n \times 76) \div 10$

2. Express the quotient of each pair of numbers below in simplest mixed form. Also express the relation between the numbers in the form $a = b \times n + r$

Example: 44, 6

$$44 \div 6 = \frac{44}{6} = 7\frac{2}{6} = 7\frac{1}{3}$$

$$44 = (6 \times 7) + 2$$

a. 46, 7 $[6\frac{4}{7};\ 46 = (7 \times 6) + 4]$

b. 98, 13 $[7\frac{7}{13};\ 98 = (13 \times 7) + 7]$

c. 68, 12 $[5\frac{2}{3};\ 68 = (12 \times 5) + 8]$

d. 55, 8 $[6\frac{7}{8};\ 55 = (8 \times 6) + 7]$

e. 104, 13 $(8;\ 104 = 13 \times 8)$

f. 365, 7 $[52\frac{1}{7};\ 365 = (7 \times 52) + 1]$

g. 130, 16 $[8\frac{1}{8};\ 130 = (16 \times 8) + 2)$

3. The perimeter of a square is 17 inches. What is the length of one side? $(4\frac{1}{4}$ in.)

4. Ann's mother divided a quart (32 ounces) of lemonade among 5 children. How many ounces did each child get? $(6\frac{2}{5}$ oz.)

5. In Nevada, Joan's family drove 420 miles in 8 hours. What was their rate in miles per hour? $(52\frac{1}{2}$ m.p.h.)

6. **BRAINTWISTER**. Suppose m, n, and p are counting numbers.

   a. Translate this sentence into the language of fractions.

$$(m + n) \div p = (m \div p) + (n \div p).$$

$$\left(\frac{m+n}{p} = \frac{m}{p} + \frac{n}{p}\right)$$

   b. In grade 4, we found that the sentence was true if p was a common factor of m and n as in

$$96 \div 8 = 80 \div 8 + 16 \div 8.$$

   What did we call this property? (Distributive)

   c. If we use division of rational numbers, is the sentence in (a) true for any counting numbers m, n, and p? (yes)

   Show why or why not.

$$\begin{aligned}(m+n) \div p &= \frac{m+n}{1} \div \frac{p}{1}\\ &= \frac{m+n}{1} \times \frac{1}{p}\\ &= \frac{m+n}{p}\\ &= \frac{m}{p} + \frac{n}{p})\end{aligned}$$

EXTENDING THE SYSTEM OF FRACTIONS

Exploration

You have seen that $\frac{7}{3}$ is the quotient $7 \div 3$. Now look at the quotient $3 \div .4$

Even though $.4$ is not a whole number, it is convenient to write the division expression $3 \div .4$ as the fraction $\frac{3}{.4}$.

This is just like inventing a new word by telling what it is to mean. We simply agree that the new symbol $\frac{3}{.4}$ names the result of operating on 3 and $.4$ by division.

The sentence

$$\frac{3}{.4} = 3 \div .4$$

tells the meaning of $\frac{3}{.4}$.

We will use this meaning for all numerals of the form $\frac{a}{b}$ with rational numbers for a and b $(b \neq 0)$.

$$\frac{6.2}{2.7} = 6.2 \div 2.7$$

$$\frac{3.4}{.6} = 3.4 \div .6.$$

This meaning for fractions agrees with what we already know about fractions using whole numbers.

$$\frac{7}{4} = 7 \div 4$$

If our "new" fractions did not have the same properties as our "old" fractions, there would be no reason to use such symbols for quotients. In fact, it would be confusing to do so. We can show that these properties are still true, and we can use them in division problems. As examples we give two of these properties of all fractions and show why they are true.

For a fraction with whole number numerator and denominator you know that

$$\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{3 \times 3}{4 \times 3} = \ldots.$$

We might express this property as

Property (I)

$$\frac{a}{b} = \frac{a \times m}{b \times m},$$

when a, b, and m are whole numbers $(b \neq 0, m \neq 0)$.

Test it, for rational numbers, in this example:

Is it true that

$$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}} ?$$

$$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$$

$$\frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}} = \frac{\frac{3}{8}}{\frac{3}{12}} = \frac{3}{8} \div \frac{3}{12} = \frac{3}{8} \times \frac{12}{3} = \frac{3 \times 12}{8 \times 3} = \frac{12}{8} = \frac{3}{2}$$

So $$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}}$$

Look at this example in another way to see whether this property will always be true.

Does $$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}} ?$$

Suppose $$\frac{\frac{1}{2}}{\frac{1}{3}} = n$$

Then $$n = \frac{1}{2} \div \frac{1}{3}$$ (Fraction symbol indicates division)

$$n \times \frac{1}{3} = \frac{1}{2}$$ (We can rewrite a division sentence as a multiplication sentence)

$$(n \times \frac{1}{3}) \times \frac{3}{4} = \frac{1}{2} \times \frac{3}{4}$$ (Since $(n \times \frac{1}{3})$ and $\frac{1}{2}$ name the same number)

$$n \times (\frac{1}{3} \times \frac{3}{4}) = \frac{1}{2} \times \frac{3}{4}$$ (Associative Property)

$$n = (\frac{1}{2} \times \frac{3}{4}) \div (\frac{1}{3} \times \frac{3}{4})$$ (We can rewrite a multiplication sentence as a division sentence)

$$n = \frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}}$$ (A quotient can be rewritten as a fraction)

So $$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{3} \times \frac{3}{4}}$$

So Property (I) was true in this case.

This same way of thinking can be used to show that Property I is always true if a, b, and m are rational numbers and b and m are not 0.

Use Property I to find other fraction names for these numbers. Multiply the numerator and denominator by the number suggested for m.

1. $\frac{\frac{3}{8}}{\frac{5}{4}}$ $m = 8\left(\frac{\frac{3}{8}\times 8}{\frac{5}{4}\times 8} = \frac{3}{10}\right)$ 2. $\frac{\frac{7}{3}}{\frac{5}{6}}$ $m = 6\left(\frac{\frac{7}{3}\times 6}{\frac{5}{6}\times 6} = \frac{14}{5}\right)$

3. $\frac{1.5}{2.7}$ $m = 10\left(\frac{1.5\times 10}{2.7\times 10} = \frac{15}{27}\right)$ 4. $\frac{700}{800}$ $m = \frac{1}{100}\left(\frac{700\times\frac{1}{100}}{800\times\frac{1}{100}} = \frac{7}{8}\right)$

5. To rename $\frac{\frac{1}{2}}{\frac{1}{3}}$ as a fraction with whole number numerator and denominator, what number should you use for m? Show that your choice for m is correct. ($m = 6$, or any multiple of 6. $\frac{\frac{1}{2}\times 6}{\frac{1}{3}\times 6} = \frac{3}{2}$)

6. To rename $\frac{\frac{3}{8}}{\frac{5}{4}}$ so the denominator of the fraction name will be 1, what number should you use for m? Show that your choice for m is correct. ($m = \frac{4}{5}$. $\frac{\frac{3}{8}\times\frac{4}{5}}{\frac{5}{4}\times\frac{4}{5}} = \frac{\frac{12}{40}}{1}$)

7. Rename the number in exercise 1 by dividing $\frac{3}{8}$ by $\frac{5}{4}$.
($\frac{3}{8} \div \frac{5}{4} = \frac{3}{8} \times \frac{4}{5} = \frac{12}{40}$ or $\frac{3}{10}$)

8. Rename the number in exercise 2 by dividing $\frac{7}{3}$ by $\frac{5}{6}$.
($\frac{7}{3} \div \frac{5}{6} = \frac{7}{3} \times \frac{6}{5} = \frac{14}{5}$)

9. Do your answers for exercises 1 and 7 name the same number? (yes) What about your answers for exercises 2 and 8?
(The answers are the same.)

Exercise Set 16

Use Property I $\left(\frac{a}{b} = \frac{a \times m}{b \times m}\right)$ to rename each number by a fraction with whole number numerator and denominator.

1. $\dfrac{\frac{5}{8}}{\frac{3}{4}} \left(\frac{5}{6}\right)$

2. $\dfrac{\frac{7}{5}}{\frac{8}{3}} \left(\frac{21}{40}\right)$

3. $\dfrac{\frac{3}{2}}{\frac{7}{10}} \left(\frac{15}{7}\right)$

4. $\dfrac{6}{3\frac{1}{7}} \left(\frac{42}{22} \text{ or } \frac{21}{11}\right)$

5. $\dfrac{.9}{.07} \left(\frac{90}{7}\right)$

6. $\dfrac{1.25}{2.3}$ $\left(\frac{125}{230} \text{ or } \frac{25}{46}\right)$

Use Property I $\left(\frac{a}{b} = \frac{a \times m}{b \times m}\right)$ to show that each of these sentences is true.

7. $\dfrac{\frac{3}{10}}{\frac{7}{10}} = \frac{3}{7} \left(\dfrac{\frac{3}{10} \times 10}{\frac{7}{10} \times 10} = \frac{3}{7}\right)$

8. $\dfrac{.45}{.7} = \frac{45}{70} \left(\dfrac{\frac{45}{100} \times 100}{\frac{7}{10} \times 100} = \frac{45}{70}\right)$

9. $\dfrac{2.47}{.6} = \frac{24.7}{6} \left(\dfrac{\frac{247}{100} \times 10}{\frac{6}{10} \times 10} = \frac{24.7}{6}\right)$

10. $\dfrac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2} \left(\dfrac{\frac{1}{2} \times 6}{\frac{1}{3} \times 6} = \frac{3}{2}\right)$

11. $\dfrac{4}{5} = \dfrac{\frac{4}{5}}{1} \left(\dfrac{\frac{4}{5} \times 5}{1 \times 5} = \frac{4}{5}\right)$

12. $\dfrac{\frac{2}{5}}{.67} = \frac{40}{67} \left(\dfrac{\frac{2}{5} \times 100}{\frac{67}{100} \times 100} = \frac{40}{67}\right)$

13. $\dfrac{.4}{2\frac{3}{5}} = \frac{2}{13} \left(\dfrac{\frac{4}{10} \times 10}{\frac{13}{5} \times 10} = \frac{4}{26} = \frac{2}{13}\right)$

14. $\dfrac{2\frac{1}{7}}{2\frac{1}{7}} = 1 \left(\dfrac{\frac{15}{7} \times 7}{\frac{15}{7} \times 7} = \frac{15}{15} = 1\right)$

Exploration

You have seen that the property $\frac{a}{b} = \frac{a \times m}{b \times m}$ is true when a, b, and m are rational numbers (not 0.)

Here is another property of "whole number fractions"

$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5}, \qquad \frac{7}{8} \times \frac{3}{4} = \frac{7 \times 3}{8 \times 4}.$$

We shall call this property "Property II"

Property (II) We write this property as:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \quad (\text{If } b \neq 0 \text{ and } d \neq 0)$$

Is this property still true if a, b, c, and d are rational numbers but not necessarily whole numbers?

As an illustration, is this sentence true?

$$\frac{\frac{1}{2}}{\frac{2}{3}} \times \frac{3}{\frac{1}{5}} = \frac{\frac{1}{2} \times 3}{\frac{2}{3} \times \frac{1}{5}} = \frac{\frac{3}{2}}{\frac{2}{15}}$$

To see that the sentence is correct, we remember:

$$\frac{\frac{1}{2}}{\frac{2}{3}} = \frac{1}{2} \div \frac{2}{3} = \frac{1}{2} \times \frac{3}{2}$$

$$\frac{3}{\frac{1}{5}} = 3 \div \frac{1}{5} = 3 \times 5$$

Then

$$\frac{\frac{1}{2}}{\frac{2}{3}} \times \frac{3}{\frac{1}{5}} = (\tfrac{1}{2} \times \tfrac{3}{2}) \times (3 \times 5)$$

$$= (\tfrac{1}{2} \times 3) \times (\tfrac{3}{2} \times 5) \qquad \text{(Why?)}$$

(associative and Commutative Property)

$$= \frac{3}{2} \times \frac{15}{2}$$

$$= \frac{3}{2} \div \frac{2}{15} \qquad \text{(Why?)}$$

$\left(\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}\right)$

$$= \frac{\frac{3}{2}}{\frac{2}{15}}$$

This way of thinking can show that Property II

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

is true if a, b, c and d are rational numbers $(b \neq 0, d \neq 0)$.

Use Properties (I) and (II) to show that each of these sentences is true:

1. $\frac{4}{5} \times \frac{\frac{1}{3}}{\frac{1}{6}} = \frac{8}{5}$ $\left(\frac{4}{5} \times \frac{\frac{1}{3}}{\frac{1}{6}} = \frac{4}{5} \times \frac{\frac{1}{3} \times 6}{\frac{1}{6} \times 6} = \frac{4}{5} \times \frac{2}{1} = \frac{8}{5}\right.$ or
$\left.\frac{4}{5} \times \frac{\frac{1}{3}}{\frac{1}{6}} = \frac{4 \times \frac{1}{3}}{5 \times \frac{1}{6}} = \frac{\frac{4}{3}}{\frac{5}{6}} = \frac{\frac{4}{3} \times 6}{\frac{5}{6} \times 6} = \frac{8}{5}\right)$

2. $\frac{.6}{.7} \times \frac{.3}{.2} = \frac{9}{7}$ $\left(\frac{.6}{.7} \times \frac{.3}{.2} = \frac{.6 \times 10}{.7 \times 10} \times \frac{.3 \times 10}{.2 \times 10} = \frac{6}{7} \times \frac{3}{2} = \frac{18}{14} \text{ or } \frac{9}{7}\right.$
or $\left.\frac{.6}{.7} \times \frac{.3}{.2} = \frac{.18}{.14} = \frac{.18 \times 100}{.14 \times 100} = \frac{18}{14} \text{ or } \frac{9}{7}\right)$

3. $3 \times \frac{\frac{2}{5}}{\frac{1}{4}} = 4\frac{4}{5}$ $\left(3 \times \frac{\frac{2}{5}}{\frac{1}{4}} = \frac{3}{1} \times \frac{\frac{2}{5} \times 20}{\frac{1}{4} \times 20} = \frac{3}{1} \times \frac{8}{5} = \frac{24}{5} = 4\frac{4}{5}\right.$
or $\left.\frac{3 \times \frac{2}{5}}{1 \times \frac{1}{4}} = \frac{\frac{6}{5}}{\frac{1}{4}} = \frac{\frac{6}{5} \times 20}{\frac{1}{4} \times 20} = \frac{24}{5} = 4\frac{4}{5}\right)$

Exercise Set 17

1. Since every fraction with rational numerator and denominator names a rational number, it must have a simplest fraction name and a simplest mixed form

Find each of these for the fractions below.

Example: $\frac{5.5}{1.7}$

$\frac{5.5}{1.7} = \frac{5.5 \times 10}{1.7 \times 10} = \frac{55}{17}$ (simplest fraction name)

$\frac{55}{17} = 3\frac{4}{17}$ (simplest mixed form)

a. $\frac{6.3}{5}$ $\left(\frac{63}{50},\ 1\frac{13}{50}\right)$

b. $\frac{\frac{3}{4}}{\frac{2}{3}}$ $\left(\frac{9}{8},\ 1\frac{1}{8}\right)$

c. $\frac{1.8}{.5}$ $\left(\frac{18}{5},\ 3\frac{3}{5}\right)$

d. $\frac{2\frac{1}{2}}{.6}$ $\left(\frac{25}{6},\ 4\frac{1}{6}\right)$

2. Because the middle "bar" in $\frac{\frac{3}{4}}{\frac{2}{3}}$ means division we can write $\frac{\frac{3}{4}}{\frac{2}{3}} = \frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$. Use this method to find simplest forms for the following:

a. $\frac{\frac{1}{2}}{\frac{1}{3}}$ $\left(\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times 3 = \frac{3}{2}\right)$

b. $\frac{3}{\frac{1}{4}}$ $\left(3 \div \frac{1}{4} = 3 \times 4 = 12\right)$

c. $\frac{1\frac{1}{2}}{\frac{2}{3}}$ $\left(\frac{3}{2} \div \frac{2}{3} = \frac{3}{2} \times \frac{3}{2} = \frac{9}{4}\right)$

d. $\frac{\frac{5}{8}}{\frac{8}{5}}$ $\left(\frac{5}{8} \div \frac{8}{5} = \frac{5}{8} \times \frac{5}{8} = \frac{25}{64}\right)$

V. DIVISION OF RATIONAL NUMBERS NAMED BY DECIMALS
EXTENDING THE DIVISION PROCESS
ESTIMATING RATIONAL NUMBERS USING DECIMALS

Objectives: To find a method for computing quotients of rational numbers, using decimal numerals.

To use the division process to find decimal names for quotients.

To determine whether or not a rational number named by a fraction has a decimal name; to find a decimal estimate for those rational numbers for which we do not as yet have decimal names.

Vocabulary: Decimal, fraction form

In the section, "Division of Rational Numbers Named by Decimals", the ideas developed in the two previous sections are applied. A division example written with decimal numerals is first rewritten as a fraction. Then the property $\frac{a}{b} = \frac{a \times m}{b \times m}$ is applied, with m a power of 10 such that $(a \times m)$ and $(b \times m)$ are whole numbers. The division algorism is then used to obtain an answer which names a whole number or which is a mixed form.

This procedure makes possible the use of the division algorism when the dividend and divisor are named by decimal numerals, but may provide a quotient named by a mixed form rather than a decimal numeral. (Recall that "decimal numeral" means a numeral in base ten. 548 is a decimal numeral, as are 5480, 5.48, 54.8, etc.) In the section "Extending the Division Process", the idea is developed that renaming the dividend makes it possible to continue the division algorism used with whole numbers and find the decimal numeral for the quotient if it has a decimal name. In the examples in the Exploration, the decimal for the quotient $187 \div 25$ is found by first renaming 187 as 187.00. The decimal for the quotient $87.4 \div 25$ is found by first renaming 87.4 as 87.400. The decimal for $12.68 \div .4$ is found by rewriting the quotient expression as $\frac{12.68}{.4}$, then renaming the quotient by $\frac{126.8}{4}$.

In working through the Exploration with the class, it may be necessary to emphasize the importance of arranging written work carefully so that the place value of a digit in a product or a remainder can be determined easily.

In Exercise Set 18, the first four parts of exercise 1 should be discussed so the pupils note these relations:

a. $231 \div 15 = 231.0 \div 15 = 15.4$ (By the division algorism)

b. $$23.1 \div .15 = \frac{23.1}{.15} = \frac{231 \times .1}{15 \times .01}$$

$$= \frac{231}{15} \times \frac{.1}{.01}$$

$$= \frac{231}{15} \times \frac{.1 \times 100}{.01 \times 100}$$

$$= \frac{231}{15} \times \frac{10}{1} \quad \text{From a,} \quad \frac{231}{15} = 15.4.$$

$$= 15.4 \times 10 = 154$$

c. $$.231 \div 1.5 = \frac{.231}{1.5} = \frac{231 \times .001}{15 \times .1}$$

$$= \frac{231}{15} \times \frac{.001}{.1}$$

$$= 15.4 \times \frac{.001 \times 1000}{.1 \times 1000}$$

$$= 15.4 \times \frac{1}{100}$$

$$= .154$$

d. $$2.31 \div 15 = \frac{2.31}{15} = \frac{231 \times .01}{15 \times 1}$$

$$= \frac{231}{15} \times \frac{.01}{1}$$

$$= 15.4 \times \frac{1}{100}$$

$$= .154$$

In the section "Estimating Rational Numbers Using Decimals" we consider decimals for rational numbers for which we do not as yet have decimal names. Such numbers were considered briefly in Chapter 2. In the Exploration in this section, a method is developed for deciding whether a rational number has a decimal name. This is done by examining the complete factorization of denominator of its simplest fraction name. If the denominator has any prime factor other than 2 or 5 (or 2 and 5), then it has no decimal name. This is a consequence of the decimal (base ten) system of notation. In this system, the fraction form of any decimal has some power of 10 as denominator, and powers of 10 have only 2 and 5 as prime factors.

For rational numbers for which we do not as yet have decimal numerals, as close approximations as are desired may be found, by continuing the division process. In the Exploration it is shown that $\frac{11}{9}$ differs from 1.22 by only $\frac{2}{900}$ and from 1.222 by only $\frac{2}{9000}$.

The latter fact may be verified as follows:

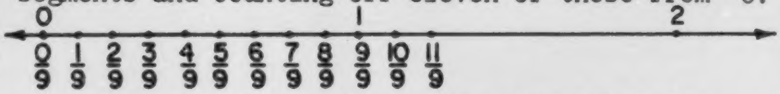
$$
\begin{aligned}
1.222 + \frac{2}{9000} & \\
&= \frac{1222}{1000} + \frac{2}{9000} \\
&= \frac{10998}{9000} + \frac{2}{9000} \\
&= \frac{11000}{9000} \\
&= \frac{11}{9}
\end{aligned}
$$

Because of the convenience of decimals for comparing rational numbers and for computing with them, approximations are commonly used.

In later grades the concept of the infinite decimal will be introduced. For example, the number $\frac{1}{3}$ will be named by the decimal ".3333...", the three dots indicating that the digit 3 is repeated indefinitely. When the concept of the infinite decimal is developed, every rational number will have an infinite decimal name.

There is a way to think geometrically about the division process, and to picture the quotients named by decimals obtained at successive stages in the process. Such a picture can help us see what the estimates mean and how to use them.

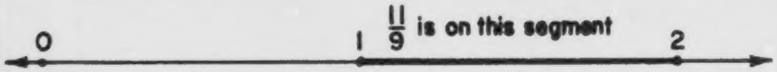
Imagine the rational numbers as points of a line. The fraction names give us directions for locating the point in terms of the points 0 and 1. For example, the point $\frac{11}{9}$ is located by separating the unit segment into nine congruent segments and counting off eleven of these from 0.



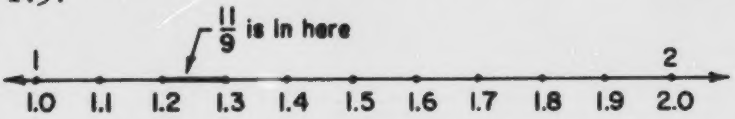
Now let's think about the information the division process gives us about $\frac{11}{9}$. We wish to locate the point for $\frac{11}{9}$ on a number line scaled only in tenths, hundredths, thousandths, etc.

First we found $\frac{11}{9} = 1 + \frac{2}{9}$

Since $\frac{2}{9}$ is less than 1, we can conclude that $\frac{11}{9} > 1$ and $\frac{11}{9} < 2$. We have located $\frac{11}{9}$ between units.



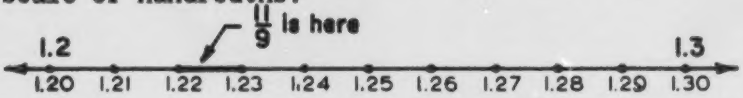
Second we found that $\frac{11}{9} = 1.2 + \frac{.2}{9} = 1.2 + (\frac{2}{9} \times .1)$. The number $\frac{2}{9} \times .1 < .1$. This means that $\frac{11}{9} > 1.2$ and $\frac{11}{9} < 1.3$. Imagine the part of the line between the points 1 and 2 magnified greatly. What we have done so far by the division process is to locate $\frac{11}{9}$ between two scale points of a scale of tenths, the points 1.2 and 1.3. It is on the segment which has 1.2 and 1.3 as endpoints, but it is not either endpoint, since $\frac{11}{9} > 1.2$ and $\frac{11}{9} < 1.3$.



Next we found that $\frac{11}{9} = 1.22 + \frac{.02}{9} =$ $1.22 + (\frac{2}{9} \times .01)$.

Since $\frac{2}{9} \times .01 < .01$ we know that $\frac{11}{9} > 1.22$ and $\frac{11}{9} < 1.23$.
We have located $\frac{11}{9}$ between two scale points of a number line scaled in hundredths. It is on the segment with endpoints 1.22 and 1.23, but it is not either endpoint.

Imagine the part of the line between the points 1.2 and 1.3 magnified again, and divided into ten congruent segments to show a scale of hundredths.



Suppose we divided the shaded segment with endpoints 1.22 and 1.23 into ten congruent segments, to construct a scale of thousandths. The point $\frac{11}{9}$ would fall on one of these segments. Our division process shows that it would fall on the segment with endpoints 1.222 and 1.223, because

$$\frac{11}{9} = 1.222 + (\frac{2}{9} \times .001).$$

It would not be either endpoint, because $\frac{11}{9} > 1.222$ and $\frac{11}{9} < 1.223$.

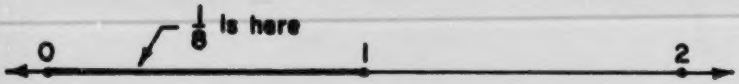
Thus, the successive stages of the division process serve to locate the point for $\frac{11}{9}$ on successively smaller segments of the number line. By continuing the process we can locate it on as small a segment as we wish.

One more example is necessary: $\frac{1}{8}$

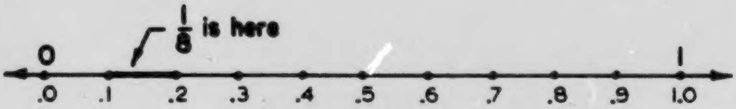
```
     0.             .1          .12            .125
 8 ) 1.      8 ) 1.0     8 ) 1.00      8 ) 1.000
                 8           8              8
                 -           -              -
                 2           20             20
                             16             16
                             --             --
                              4             40
                                            40
                                            --
```

The information given by the successive stages of the division process may be pictured, with successive magnifications, as shown below.

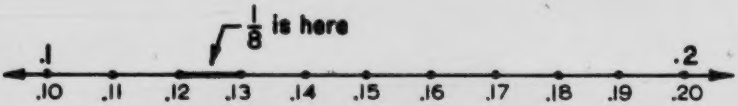
$$\frac{1}{8} > 0 \quad \text{and} \quad \frac{1}{8} < 1$$


1/8 is here
0
1
2

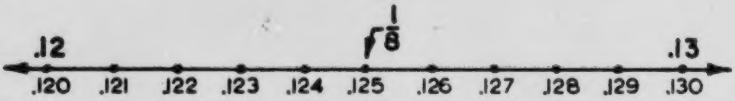
$$\frac{1}{8} > .1 \quad \text{and} \quad \frac{1}{8} < .2$$


1/8 is here
0
1
.0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0

$$\frac{1}{8} > .12 \quad \text{and} \quad \frac{1}{8} < .13$$


1/8 is here
.1
.2
.10
.11
.12
.13
.14
.15
.16
.17
.18
.19
.20

$$\frac{1}{8} = .125$$


1/8
.12
.13
.120
.121
.122
.123
.124
.125
.126
.127
.128
.129
.130

Since $\frac{1}{8} = .125$, the decimal numeral specifies the exact location of the point $\frac{1}{8}$. We see that it is one of the division points of the scale of thousandths.

Here is a summary of some of the properties of decimal numerals and their geometric interpretations.

1) A rational number whose simplest fraction name has a denominator with prime factors only 2, 5, or 2 and 5, has a decimal numeral.

2) The division process can be used to translate fractions into decimals. If the rational number named by the fraction has no decimal name, then the process gives decimal approximations in tenths, hundredths, thousandths, etc.

3) If we think of rational numbers as points on a line, then those numbers with decimal numerals are the division points obtained by dividing the unit segments into tenths, hundredths, thousandths, etc. The rational numbers which do not have decimal numerals are the points which always fall between the division points, no matter how far the division process is continued.

DIVISION OF RATIONAL NUMBERS NAMED BY DECIMALS

Exploration

Suppose that we are using only whole numbers and have a division problem like this:

Divide 1267 by 240.

What do we find? We have a division process for finding a relation between 1267 and 240. If 1267 is a multiple of 240, the process will give the decimal numeral for the whole number $1267 \div 240$. If 1267 is not a multiple of 240, the process will find what multiple of 240 less than 1267 is closest to 1267. We show our process in this way.

$$\begin{array}{r@{\,}l} & 5 \\ 240 & \overline{)\,1267} \\ & \underline{\,1200} \\ & 67 \end{array}$$

The relation is expressed by this sentence:

$$1267 = (240 \times 5) + 67.$$

We can not write a division sentence because no whole number n makes these sentences true:

$$1267 = 240 \times n$$

$$1267 \div 240 = n$$

If we use rational numbers then we can write such sentences. They will be

$$1267 = 240 \times (5\tfrac{67}{240}) \quad \text{or}$$

$$1267 \div 240 = 5\tfrac{67}{240}.$$

This is the usual way in which the instructions

Divide 1267 by 240

are followed. We use the division process to find a name for the rational number $1267 \div 240$. The division process as you have learned it gives a mixed form for the quotient.

Use the division process to find a mixed form for each of these:

1. $187 \div 24$ $\left(7\frac{19}{24}\right)$
2. $277 \div 31$ $\left(8\frac{29}{31}\right)$
3. $207 \div 23$ (9)
4. $875 \div 43$ $\left(20\frac{15}{43}\right)$

Can we use the division process to get mixed form names for quotients of any rational numbers whose decimals are given? Suppose our problem is

$$12.67 \div 2.4 = n$$

(n is to be named by a mixed form.)

The division sentence $12.67 \div 2.4 = n$ might be written as

$$\frac{12.67}{2.4} = n$$

Then we could write

$$\frac{12.67}{2.4} = \frac{12.67 \times 100}{2.4 \times 100} = \frac{1267}{240} = 1267 \div 240 = 5\frac{67}{240}$$

We know how to translate any fraction like $\frac{1267}{240}$ into mixed form.

The trick is to rename the number by a "whole number" fraction.

Example: $\frac{23.5}{1.74} = \frac{23.5 \times 10^2}{1.74 \times 10^2} = \frac{2350}{174}$

$$\begin{array}{r} 13 \\ 174\,\overline{)\,2350} \\ \underline{174} \\ 610 \\ \underline{522} \\ 88 \end{array} \qquad \frac{23.5}{17.4} = 13\frac{88}{174} = 13\frac{44}{87}$$

Express each of these quotients in simplest mixed form:

5. $\frac{15.6}{2.3}$ $\left(6\frac{18}{23}\right)$

6. $\frac{258.7}{23}$ $\left(11\frac{57}{230}\right)$

7. $438 \div 14.9$ $\left(29\frac{59}{149}\right)$

Many times it is convenient to express the quotient of two rational numbers as a decimal. If the numbers are named as decimals, then it is natural to compute in decimal language. One difficulty is that as yet we do not have a decimal name for every rational number. As yet we have no decimals for $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{6}$, etc. A second difficulty is that our process may not give the decimal name even when there is one. Here is an example:

$$\frac{15.6}{2} = \frac{156}{20} \qquad \begin{array}{r} 7 \\ 20\,\overline{)\,156} \\ \underline{140} \\ 16 \end{array} \qquad \frac{15.6}{2} = 7\frac{16}{20} = 7\frac{80}{100} = 7.80 \text{ or } 7.8.$$

We had to find a decimal name after finding a mixed form. The division process alone did not give us the decimal 7.8.

In the next sections we will find out how to extend the division process to find decimal names.

Exercise Set 18

1. Which are other names for the first number in each row?

a. $\frac{25.6}{.14}$; $\frac{2560}{14}$, $\frac{1280}{7}$, $\frac{2.56}{1.4}$ $\left(\frac{2560}{14}, \frac{1280}{7}\right)$

b. $\frac{13.54}{.5}$; 13.54×2, 13.54×5, $27\frac{2}{25}$ $\left(13.54 \times 2, 27\frac{2}{25}\right)$

c. $16 \div 7\frac{2}{5}$; $\frac{16}{7\frac{2}{5}}$, $\frac{80}{37}$, $\frac{.08}{.037}$ $\left(\frac{16}{7\frac{2}{5}}, \frac{80}{37}, \frac{.08}{.037}\right)$

d. $\frac{2.64}{.32}$; $\frac{264}{32}$ $8\frac{1}{4}$, $\frac{.0264}{.032}$ $\left(\frac{264}{32}, 8\frac{1}{4}\right)$

2. A rectangular region whose adjacent sides have measures 3.2 and 6.7 is separated into 16 smaller congruent rectangular regions. What is the measure of each of the smaller regions? (1.34 sq. in.)

3. a. Which of these numbers is the greater,

$$117 \div 36 \quad \text{or} \quad 113.4 \div 35?$$ ($117 \div 36$)

b. How much greater? (0.01)

4. If a plane travels 2750 miles in 5.5 hours, how far will it travel in 3 hours? (1500 mi.)

EXTENDING THE DIVISION PROCESS

Exploration

Consider the quotient $18.7 \div 2.5$. We know that

$$18.7 \div 2.5 = \frac{18.7}{2.5} = \frac{187}{25}$$

We can find by the division process

$$\begin{array}{r} 7 \\ 25\,\overline{)\,187} \\ \underline{175} \\ 12 \end{array}$$

that

$$\frac{187}{25} = 7\frac{12}{25}$$

We know that

$$7\frac{12}{25} = 7\frac{48}{100} = 7.48$$

Now we know that $18.7 \div 2.5 = 7.48$, but the division process did not give us this name for the answer. The question is this: Can we extend the division process so that it gives us this decimal name?

To get an idea about the answer to this question, study the following quotients. Show that each answer is correct.

1. $18{,}700 \div 25 =$ 748

2. $1870 \div 25 = \frac{1870}{25} = 74\frac{20}{25} = 74\frac{80}{100} =$ 74.8

3. $187 \div 25 = \frac{187}{25} = 7\frac{12}{25} = 7\frac{48}{100} =$ 7.48

4. $18.7 \div 25 = \frac{18.7}{25} = \frac{187}{250} = \frac{748}{1000} =$ $.748$

P389

5. What do you think is the decimal for

$$187{,}000 \div 25?$$

for $$1.87 \div 25?$$

How can you explain the fact that the answers have the same digits and differ only in the position of the decimal point? Consider exercises 1 and 2, to see how you can use the answers for exercise 1 to find the answer for exercise 2.

$$\frac{18{,}700}{25} = 748$$

$$\frac{1870}{25} = \frac{1870 \times 10}{25 \times 10}$$

$$= \frac{18{,}700 \times 1}{25 \times 10}$$

$$= \frac{18{,}700}{25} \times \frac{1}{10}$$

$$= 748 \times \frac{1}{10}$$

$$= 74.8$$

Now consider the division process for 18.7 ÷ 25 again. Think of 187 as 187.00 and divide as usual.

6. $18.7 \div 2.5 = \frac{187}{25}$.

---

a.

$$\begin{array}{rr} & 7. \\ 25 & \overline{)187.00} \\ & \underline{175} \\ & 12 \end{array}$$

The 7 is 7 ones, so put a decimal point after 7. The remainder is 12 ones.

b.

$$\begin{array}{rr} & 7.4 \\ 25 & \overline{)187.00} \\ & \underline{175} \\ & 12\ 0 \end{array}$$

Think of 12 ones as 120 tenths. 120 ÷ 25 is about 4, so 120 tenths ÷ 25 is about 4 tenths. Write the 4 in tenths' place.

c.

$$\begin{array}{rr} & 7.4 \\ 25 & \overline{)187.00} \\ & \underline{175} \\ & 12\ 0 \\ & \underline{10\ 0} \\ & 2\ 0 \end{array}$$

25 x 4 tenths is 100 tenths. The remainder is 20 tenths.

d.

$$\begin{array}{rr} & 7.48 \\ 25 & \overline{)187.00} \\ & \underline{175} \\ & 12\ 0 \\ & \underline{10\ 0} \\ & 2\ 00 \\ & \underline{2\ 00} \end{array}$$

Think of 20 tenths as 200 hundredths.
25 x 8 hundredths = 200 hundredths.
Write the 8 in hundredths' place.

18.7 ÷ 2.5 = 7.48

You see that you can find the digits in the answer just as though you were dividing 18700 by 25. If you place the decimal point carefully you do not need to think again of tenths and hundredths.

Look at another example:

7. Find the decimal for $87.4 \div 25$.

a.
$$
\begin{array}{rl}
 & \;\;3. \\
25 & \big)\overline{87.4} \\
 & \underline{75} \\
 & 12\ 4
\end{array}
$$

The remainder is really 12.4

The division shows that

$$87.4 = (25 \times 3) + 12.4$$

Since $\frac{12.4}{25} = \frac{124}{250}$ or $\frac{124}{25} \times \frac{1}{10}$,

place the next digit in tenths' place.

b.
$$
\begin{array}{rl}
 & \;\;3.4 \\
25 & \big)\overline{87.4} \\
 & \underline{75} \\
 & 12\ 4 \\
 & \underline{10\ 0} \\
 & \;\;2\ 4
\end{array}
$$

25 × 4 tenths = 100 tenths.

The remainder is really 2.4

Think of 2.4 as 2.40, or 240 hundredths, and continue the process.

c.
$$
\begin{array}{rl}
 & \;\;3.496 \\
25 & \big)\overline{87.400} \\
 & \underline{75} \\
 & 12\ 4 \\
 & \underline{10\ 0} \\
 & \;\;2\ 40 \\
 & \;\;\underline{2\ 25} \\
 & \;\;\;\;\;150 \\
 & \;\;\;\;\;\underline{150}
\end{array}
$$

$$87.4 \div 25 = 3.496$$

8. How does this process compare with what you would have done to find the decimal numeral for 87,400 ÷ 25?
(It is the same except for the decimal point)

9. Exercise 7 shows that you can use this extended process without rewriting the problem so that both numerator and denominator of the fraction are whole numbers. Rename the fraction so the denominator is a whole number.

Look at this example:

Find the decimal for 12.68 ÷ .4

a. $12.68 \div .4 = \frac{12.68}{.4} = \frac{126.8}{4}$

Now proceed as before:

b. $$\begin{array}{r} 31.7 \\ 4\;\overline{)\,126.8} \\ \underline{12} \\ 6 \\ \underline{4} \\ 2\;8 \\ \underline{2\;8} \end{array}$$

$12.68 \div .4 = 31.7$

10. You know that $12.68 \div .4 = 31.7$ is a true sentence if $.4 \times 31.7 = 12.68$ is a true sentence. Use the multiplication process to show that this sentence is true.

11. Use multiplication to check the answers for exercises 6 and 7.

Exercise Set 19

1. Find decimal numerals for these quotients.

a. 231 ÷ 15 (15.4)
b. 23.1 ÷ .15 (154)
c. .231 ÷ 1.5 (0.154)
d. 2.31 ÷ 15 (0.154)

e. 5.85 ÷ 7.5 (0.78)
f. 5850 ÷ 7.5 (780)
g. .585 ÷ 75 (0.0078)
h. 58.5 ÷ .75 (78)

2. Rename these numbers so the denominator of the fraction is a whole number.

a. $\frac{73.6}{.25}$ $\left(\frac{7360}{25}\right)$
b. $\frac{.097}{3.265}$ $\left(\frac{97}{3265}\right)$
c. $\frac{38.92}{5.1}$ $\left(\frac{389.2}{51}\right)$
d. $\frac{685}{8.2}$ $\left(\frac{6850}{82}\right)$

e. $\frac{390}{.127}$ $\left(\frac{390,000}{127}\right)$
f. $\frac{.063}{1.85}$ $\left(\frac{6.3}{185}\right)$
g. $\frac{649}{.36}$ $\left(\frac{64,900}{36}\right)$
h. $\frac{1.267}{5.9}$ $\left(\frac{12.67}{59}\right)$

3. Find decimal names for these quotients. Check by multiplication.

a. 1008 ÷ .6 (16,80 ; .6 × 1680 = 1008)
b. 213.9 ÷ 3.75 (57.04; 3.75 × 57.04 = 213.9)
c. 646 ÷ 6.8 (95.0; 6.8 × 95 = 646)
d. 30.94 ÷ 2.6 (11.9; 2.6 × 11.9 = 30.94)

4. Use the sentence $18.7 \div 2.5 = 7.48$ to write the decimal numeral for each of these quotients:

   a. $187 \div 25$ (7.48) b. $18700 \div 250$ (74.8) c. $18.7 \div .25$ (74.8)

5. Use $\frac{9}{25} = .36$ to write the decimal numeral for each of these.

   a. $90 \div 25$ (3.6) b. $9 \div 250$ (.036) c. $900 \div 25$ (36)

6. Make each sentence true by placing the decimal point and writing any needed zeros in the numeral for the second factor shown.

   a. $2.63 \times 31 = 8.153$ (3.1)

   b. $26.3 \times 31 = 815.3$ (Nothing needs to be added)

   c. $.263 \times 31 = 8.153$ (Nothing needs to be added)

   Express the answer to each of the following problems as a decimal numeral.

7. If 1 centimeter were exactly .4 inches, what would the measure of an inch be in centimeters? (2.5 centimeters)

8. A car used 10.5 gallons of gasoline in traveling 163.8 miles. How many miles did it travel per gallon of gasoline? (15.6 miles per gallon of gasoline.)

9. How long will it take to travel 144 miles at 32 miles per hour? (4.5 hrs)

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ESTIMATING RATIONAL NUMBERS USING DECIMALS

Exploration

Why is it that some rational numbers have no decimal names?

1. Consider these decimals. .7, 2.85, .037, 18.279. What is the denominator of the fraction form of each decimal? (10, 100, 1000, 1000)

   Since the decimal system has ten as its base, only 10, or 10^2, or 10^3, or some other power of 10 can be the denominator of the fraction form of a decimal.

2. Write the complete factorization of 10^1, of 10^2, of 10^3. What different numbers are prime factors of any power of 10?
   ($10 = 2 \times 5$, $10^2 = 2^2 \times 5^2$, $10^3 = 2^3 \times 5^3$)
   (only factors are 2 and 5)

3. Write the complete factorization for the denominator of each fraction.

   a. $\frac{5}{6}$ (2×3)
   c. $\frac{10}{21}$ (3×7)
   e. $\frac{11}{32}$ ($2 \times 2 \times 2 \times 2 \times 2$)
   b. $\frac{7}{15}$ (3×5)
   d. $\frac{9}{40}$ ($2 \times 2 \times 2 \times 5$)
   f. $\frac{13}{36}$ ($2 \times 2 \times 3 \times 3$)

4. Suppose each number named by a fraction in exercise 3 is to be renamed by another fraction with a whole number numerator and denominator. What prime factors must the denominator of the new fraction have to rename $\frac{5}{6}$? (2 and 3) to rename each of the other numbers? (b. 3, 5; c. 3, 7; d. 2^3, 5; e. 2^5; f. 2^2, 3^2) Does 10 have these factors? (only 2 and 5) Does 10^2? (only 2^2 and 5^2) Does any power of 10?

5. Which of the numbers in exercise 3 have decimal names? (d, e)

What happens if we try our extended division process on quotients which do not have decimal names?

6. Let's try $\frac{11}{9}$.

$$\frac{11}{9} = 11 \div 9$$

$$\begin{array}{r} 1 \\ 9\,\overline{)\,11} \\ \underline{9} \\ 2 \end{array} \qquad \frac{11}{9} = 1 + \frac{2}{9}$$

$$\begin{array}{r} 1.2 \\ 9\,\overline{)\,11.0} \\ \underline{9} \\ 2\,0 \\ \underline{1\,8} \\ 2 \end{array} \qquad \begin{aligned} \frac{11}{9} &= 1.2 + \frac{.2}{9} = 1.2 + \frac{2}{90} \\ &= 1.2 + \left(\frac{2}{9} \times \frac{1}{10}\right) \end{aligned}$$

$$\begin{array}{r} 1.22 \\ 9\,\overline{)\,11.00} \\ \underline{9} \\ 2\,0 \\ \underline{1\,8} \\ 2\,0 \\ \underline{1\,8} \\ 2 \end{array} \qquad \begin{aligned} \frac{11}{9} &= 1.22 + \frac{.02}{9} = 1.22 + \frac{2}{900} \\ &= 1.22 + \left(\frac{2}{9} \times \frac{1}{100}\right) \end{aligned}$$

Suppose you divide 11.000 by 9. What do you think the quotient will be?

Is there any point in continuing the division process?

We have found that

$$\frac{11}{9} = 1.22 + \frac{.02}{9} = 1.22 + \frac{2}{900}$$

During the process we found that

$$\frac{11}{9} = \underset{\underset{\text{units digit in quotient}}{\uparrow}}{1} + \frac{2}{9} \longleftarrow \text{first remainder}$$

$$\frac{11}{9} = 1.\underset{\underset{\text{tenths' digit in quotient}}{\uparrow}}{2} + \frac{.2}{9} \longleftarrow \text{second remainder}$$

If we continued we would find

$$\frac{11}{9} = 1.22\underset{\underset{\text{thousandths' digit in quotient}}{\uparrow}}{2} + \frac{.002}{9} \longleftarrow \text{fourth remainder}$$

What would the next step show? $\left(\frac{11}{9} = 1.2222 + \frac{.0002}{9}\right)$

Several more examples may help to show what can happen.

7. $5 \div 6 = \frac{5}{6}$

$$\begin{array}{r} .83 \\ 6\,\overline{)\,5.00} \\ \underline{4\;8} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

$$\frac{5}{6} = .83 + \frac{.02}{6}$$

$$= .83 + \frac{2}{600}$$

The next step would show:

$$\frac{5}{6} = .833 + \frac{.002}{6}$$

$$= .833 + \frac{2}{6000}$$

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8. $\frac{5}{11} = 5 \div 11$

$$\begin{array}{r} .454 \\ 11\,\overline{)\,5.000} \\ \underline{4\,4} \\ 60 \\ \underline{55} \\ 50 \\ \underline{44} \\ 6 \end{array}$$

We can stop now. Do you see why? (Remainder 6 occurs for second time so same pattern will repeat.)

We have found

$$\frac{5}{11} = .4 + \frac{.6}{11}$$

$$\frac{5}{11} = .45 + \frac{.05}{11}$$

$$\frac{5}{11} = .454 + \frac{.006}{11}$$

Can you write what the next step would show if you continued the division process? $\left(\frac{5}{11} = .4545 + \frac{.0005}{11}\right)$

9. $\frac{1}{8} = 1 \div 8$

$$\begin{array}{r} .125 \\ 8\,\overline{)\,1.000} \\ \underline{8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

$\frac{1}{8} = .1 + \frac{.2}{8}$

$\frac{1}{8} = .12 + \frac{.04}{8}$

$\frac{1}{8} = .125$

We can think of the division process for $\frac{1}{8}$ this way:
At the first step we found $\frac{1}{8} > .1$ and $\frac{1}{8} < .2$. We might say that we have estimated $\frac{1}{8}$ in tenths.
At the second step we found an estimate for $\frac{1}{8}$ in hundredths.

$$\frac{1}{8} > .12 \text{ and } \frac{1}{8} < .13$$

At the third step we found an estimate for $\frac{1}{8}$ in thousandths.
As soon as we subtracted we found an exact numeral for $\frac{1}{8}$ in thousandths.

$$\frac{1}{8} = .125$$

10. Can you find estimates for $\frac{11}{9}$, $\frac{5}{6}$, and $\frac{5}{11}$ in ten thousandths? ($\frac{11}{9} = 1.2222 + \frac{.0002}{9}$; $1.2222 < \frac{11}{9} < 1.2223$) ($\frac{5}{6} = .8333 + \frac{.0002}{6}$ $\frac{5}{6} > .8333$ and $\frac{5}{6} < .8334$) ($\frac{5}{11} = .4545 + \frac{.0005}{11}$; $\frac{5}{11} > .4545$ and $\frac{5}{11} < .4546$)
Will the division process ever give a decimal numeral for any of these numbers? (no)

11. a. What is the estimate for $\frac{2}{3}$ in ten-thousandths? ($.6667 > \frac{2}{3}$ $.6666 < \frac{2}{3}$)

b. Does $\frac{1}{16}$ have a decimal numeral? (yes)

c. Does $\frac{1}{40}$ have a decimal numeral? (yes)

d. Find a decimal for $\frac{8}{125}$ if there is one. (.064)

Exercise Set 20

1. Which of the following numbers have decimal names and which do not? If a number has a decimal name, find it by the division process.

a. $\frac{5}{12}$ (no decimal name) c. $\frac{57}{200}$ (.285) e. $\frac{27}{14}$ (no decimal name)

b. $\frac{7}{18}$ (No decimal name) d. $\frac{31}{40}$ (.775) f. $\frac{69}{25}$ (2.76)

2. Use the division process to estimate each number in tenths as shown in the example.

Example. $\frac{15}{7} = 15 \div 7$

```
    2.1
7 ) 15.0
    14
     1 0
       7
       3
```

$\frac{15}{7}$ is between 2.1 and 2.2

a. $\frac{19}{8}$ (between 2.3 and 2.4) d. $\frac{3}{8}$ (between 0.3 and 0.4)

b. $\frac{25}{6}$ (between 4.1 and 4.2) e. $\frac{4}{11}$ (between 0.3 and 0.4)

c. $\frac{131}{14}$ (between 9.3 and 9.4) f. $\frac{365}{7}$ (between 52.1 and 52.2)

3. Estimate each of these numbers to the nearest smaller or the exact thousandth as shown in the example.

Example: $\frac{1}{6}$

```
     .166
6 ) 1.000
     6
     40
     36
      40
      36
       4
```

Number	Nearest Smaller or Exact Thousandth	Number	Nearest Smaller or Exact Thousandth	Number	Nearest Smaller or Exact Thousandth
$\frac{1}{2}$	.500	$\frac{4}{5}$	(.800)	$\frac{2}{7}$	(.285)
$\frac{1}{3}$	(.333)	$\frac{4}{7}$	(.571)	$\frac{2}{9}$	(.222)
$\frac{1}{4}$	(.250)	$\frac{4}{9}$	(.444)	$\frac{3}{4}$	(.750)
$\frac{1}{5}$	(.200)	$\frac{5}{6}$	(.833)	$\frac{3}{5}$	(.600)
$\frac{1}{6}$	.166	$\frac{5}{7}$	(.714)	$\frac{3}{7}$	(.428)
$\frac{1}{7}$	(.142)	$\frac{5}{8}$	(.625)	$\frac{3}{8}$	(.375)
$\frac{1}{8}$	(.125)	$\frac{5}{9}$	(.555)	$\frac{6}{7}$	(.857)
$\frac{1}{9}$	(.111)	$\frac{2}{3}$	(.666)	$\frac{7}{8}$	(875)
$\frac{1}{10}$	(.100)	$\frac{2}{5}$	(.400)	$\frac{7}{9}$	(.777)
				$\frac{8}{9}$	(.888)

4. Use the table you made for exercise 3 or the division process to answer these questions. Which is larger,

a. $\frac{3}{7}$ or $\frac{4}{9}$? $\left(\frac{4}{9}\right)$

b. $\frac{5}{6}$ or $\frac{21}{25}$? $\left(\frac{21}{25}\right)$

c. $\frac{5}{9}$ or $\frac{7}{13}$? $\left(\frac{5}{9}\right)$

d. $\frac{5}{27}$ or $\frac{2}{11}$? $\left(\frac{5}{27}\right)$

e. .6254 or $\frac{5}{8}$? (.6254)

f. 1667 or $\frac{10^4}{6}$ (1667)

5. Express the following numbers as hundredths plus remainder as shown in the example.

Example: 55 ÷ 17

$$\begin{array}{r} 3.23 \\ 17\,\overline{)\,55.00} \\ \underline{51} \\ 4\,0 \\ \underline{3\,4} \\ 60 \\ \underline{51} \\ 9 \end{array}$$

$$55 \div 17 = 3.23 + \frac{.09}{17}$$

a. $2\frac{1}{2} \div .6$ $\left(4.16 + \frac{.04}{6}\right)$

b. $\frac{5}{8} \div \frac{8}{5}$ $\left(0.39 + \frac{.04}{64}\right)$

c. $\frac{365}{12}$ $\left(30.41 + \frac{.08}{12}\right)$

d. $\frac{5280}{16}$ (330.00)

e. $7\frac{5}{16} \div \frac{3}{8}$ (19.50)

f. $\frac{238,000}{186,000}$ $\left(1.27 + \frac{1.78}{186}\right)$

BRAINTWISTER

6. Which of these numbers have decimal names? If a number has a decimal name, find it by the process shown in the example.

Example: $\frac{7}{80}$

$$\frac{7}{80} = \frac{7}{5 \times 2^4} = \frac{7 \times 5^3}{5^4 \times 2^4} = \frac{7 \times 5^3}{10^4} = \frac{7 \times 125}{10^4} = \frac{875}{10^4} = .0875$$

a. $\frac{11}{40}$ $\left(\frac{11}{40} = \frac{11}{5 \times 2^3} = \frac{11 \times 5^2}{5^3 \times 2^3} = \frac{11 \times 5^2}{10^3} = \frac{275}{10^3} = .275\right)$

d. $\frac{51}{800}$ $\left(= \frac{51}{5^2 \times 2^5} = \frac{51 \times 5^3}{5^5 \times 2^5} = \frac{51 \times 5^3}{10^5} = \frac{6375}{10^5} = .06375\right)$

b. $\frac{49}{36}$ (no decimal name)

e. $\frac{13}{90}$ (no decimal name)

c. $\frac{50}{28}$ (no decimal name)

f. $\frac{3}{32}$ $\left(= \frac{3}{2^5} = \frac{3 \times 5^5}{2^5 \times 5^5} = \frac{3 \times 5^5}{10^5} = \frac{9375}{10^5} = .09375\right)$

Exercise Set 21

1. Complete the table below. The measurements given are for rectangular regions.

	Measure of One side	Measure of Adjacent Side	Perimeter	Area of Region
a.	8 feet	3 feet	22 feet	24 sq. feet
b.	(20 yds)	10 yards	(60 yds)	200 sq. yards
c.	$3\frac{3}{4}$ inches	($7\frac{1}{2}$ in.)	($22\frac{1}{2}$ in.)	$28\frac{1}{8}$ sq. inches
d.	$25\frac{3}{4}$ miles	$4\frac{1}{3}$ mile	($60\frac{1}{6}$ mi.)	($111\frac{7}{12}$ sq. mi.)
e.	5 yards	($13\frac{1}{6}$ yds)	($36\frac{1}{3}$ yds)	$65\frac{5}{6}$ sq. yards
f.	(3.4 Ft.)	25.6 feet	(58 Ft.)	87.04 sq. feet
g.	3.6 miles	(56 mi.)	(119.2 mi.)	201.6 sq. mi.
h.	7.6 inches	9.8 inches	(34.8 in.)	(74.48 sq. in.)

2. Pete made a scale drawing of the floor plan for a house. He used a length of $\frac{1}{2}$ inch to represent 5 feet. What were the lengths of the segments he drew to represent a room 18 feet by 30 feet? ($1\frac{4}{5}$" by 3")

3. Mrs. Brown mailed five boxes of cookies and four boxes of candy to friends at Christmas. Each box of cookies weighed $2\frac{1}{4}$ pounds and each box of candy weighed $\frac{3}{4}$ pounds. How many pounds of cookies and candy did she mail? ($14\frac{1}{4}$ lb.)

4. Bill is using a ruler he has made to measure lengths. He thinks it is a foot long, and he has very carefully marked it off in twelve parts of equal length. If Bill's ruler is really 13.08 inches in length, how long is one of Bill's inches? (1.09 in.)

5. When the Gray family left on a trip, the speedometer read 717.6 miles. At the end of the trip, the speedometer read 1202.1 miles.

   a. How many miles had the Grays traveled? (484.5 mi.)

   b. If the made the trip in 8.5 hours, what was their average speed? (57.0 m.p.h.)

6. The circumference of the world at the equator is 24,902.37 miles. The circumference of the world at a meridian is 24,860.44 miles. How many miles greater is the circumference at the equator than at a meridian? (41.93 mi.)

7. A salesman kept a record of the distances he traveled in one week. The distances recorded were: 76.4; 85.9; 75.3; 92.5; and 100.4 miles.

   a. What was the total number of miles the salesman traveled during the week? (430.5 mi.)

   b. What was the average distance traveled per day? (86.1 mi.)

   c. If the car averaged 17.5 miles to the gallon, how many gallons of gasoline were used during the week? (24.6 gal.)

   d. At \$ .30 per gallon, what was the cost of the gasoline? (\$7.38)

   e. Using the distance traveled during this one week as an average, find the total number of miles the salesman traveled in 50 weeks. (21,525)

   f. Using the information given in exercises c and d for average mileage and average price of gasoline, find the cost of gasoline used during a year. (\$369.00)

8. The population of Chicago is $4\frac{1}{2}$ times that of Boston. The population of New York City is $2\frac{1}{10}$ times that of Chicago. The population of New York City is how many times as great as the population of Boston? (9.45)

9. Kent has a rectangular garden $9\frac{1}{4}$ feet wide and $12\frac{2}{3}$ feet long. He wants to put a wire fence along its four sides. The wire sells for $13\frac{1}{2}$ cents a foot. How much will Kent have to pay for enough wire for his garden? (\$5.91)

10. Bill ran 100 yards in 15.6 seconds. Ray ran 75 yards in 11.8 seconds. Who ran faster? (Bill ran faster; he ran 6.41 yds. in 1 second while Ray ran 6.35 yds. in 1 second)

11. A very fast runner can run 100 yards in 9.4 seconds. Express this rate in miles per hour. (21.761 m.p.h.)

12. The moon is about 238,000 miles from the earth. The sun is about 93,000,000 miles from the earth.

    a. With the distance to the moon as a unit, what is the measure of the distance to the sun? (390.75 moon-miles)

    b. If a space ship could reach the moon in 18 hours, at that rate how many days would it take for a spaceship to reach the sun from the earth? (between 293.05 and 293.06 days)

13. The populations of the 5 largest cities in the United States are listed below. What part of the 180,000,000 people in the United States live in all of these cities? (.176)

New York	10,700,000
Los Angeles	6,700,000
Chicago	6,200,000
Philadelphia	4,300,000
Detroit	3,800,000

14. Suppose that the average cost of driving a car on a free road is 7 cents per mile. The distance from Philadelphia to Pittsburgh is 300 miles along the Pennsylvania turnpike and 360 by a free road. Suppose the toll on the turnpike is $3.50. Which is the cheaper way to travel from Philadelphia to Pittsburgh, and by how much?

(The turnpike, $0.70)

15. BRAINTWISTER

The measure of 1 inch in centimeters is about 2.54. Use this to complete the following table. Estimate measures which do not have decimal names to the nearest thousandth.

a. The measure of 1 foot in centimeters. (30.48 cm.)

b. The measure of 1 foot in meters
(1 meter = 100 centimeters) (.3048 cm.)

c. The measure of 1 yard in meters. (.9144 cm.)

d. The measure of 1 mile in kilometers (1000 meters.) (1.609344 km.)

e. The measure of 1 centimeter in inches. (.393 in.)

f. The measure of 1 meter in inches. (39.370 in.)

g. The measure of 1 kilometer in miles. (.621 mi.)

Exercise Set 22

1. Find a decimal name for each sum.

   a. 356 + 47.8 (403.8)
   b. 45 + 17.17 (62.17)
   c. 0.89 + 0.75 (1.64)
   d. 59.62 + 3.84 (63.46)
   e. 373.4 + 75.99 (449.39)
   f. 0.4 + 0.6 (1.0)

2. Find decimal names for these numbers.

   a. 56 - 9.3 (46.7)
   b. 923.1 - 74.8 (848.3)
   c. 57.48 - 36.92 (20.56)
   d. 74 - 22.45 (51.55)
   e. 37.15 - 29.8 (7.35)
   f. 469.1 - 89.74 (373.36)

3. Multiply:

   a. 0.4 by 0.2 (0.08)
   b. 38.9 by 2.67 (103.863)
   c. 760 by 4.58 (3480.8)
   d. 6846 by 5.3 (36283.8)
   e. 347.8 by 5.6 (1947.68)
   f. 7.92 by 8.9 (70.488)

4. Find a decimal for each quotient.

   a. 1144 ÷ 2.3 (497.39)
   b. 36.75 ÷ 0.49 (75.0)
   c. 142.5 ÷ 0.57 (250.0)
   d. 222.7 ÷ 6.9 (32.28)
   e. 3630 ÷ 0.30 (1947.68)
   f. 29.37 ÷ 8.9 (70.488)

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5. Perform the following operations:

a. $54.72 \div 5.7$ (9.6)
b. $56 - 9.3$ (46.7)
c. $29.9 + 63$ (92.9)
d. $220.5 \div 0.65$ (339.2)
e. 37.9×4.2 (159.18)
f. $87.4 - 39.56$ (47.84)
g. $68.08 \div 0.92$ (74.0)
h. $34 + 7.69$ (41.69)
i. 7.60×13.5 (102.6)
j. $5780 \div 8.5$ (680)
k. $7.65 - 3.6$ (4.05)
l. $979.4 + 32.85$ (1012.25)
m. 388×0.74 (287.12)
n. $227.7 \div 6.9$ (33.0)

6. Perform the following operations.

a. $87.4 - 39.56$ (47.84)
b. $68.08 \div 0.92$ (74.0)
c. $17\frac{5}{12} - 9\frac{1}{2}$ $(7\frac{11}{12})$
d. $2\frac{1}{2} \div 3\frac{5}{8}$ $(\frac{20}{29})$
e. $543.9 + 74.35$ (618.25)
f. 388×0.74 (287.12)
g. $227.7 \div 6.9$ (33.0)
h. $7\frac{1}{2} \times 3\frac{5}{6}$ $(28\frac{3}{4})$
i. $44\frac{2}{3} + 7\frac{5}{9}$ $(52\frac{2}{9})$
j. 0.3×0.2 $(.06)$

7. Perform the following operations.

a. $30.66 \div 4.2$ (7.3)
b. $92\frac{1}{4} + 73\frac{5}{6}$ $(166\frac{1}{12})$
c. $59 + 0.78$ (59.78)
d. $80 - 7\frac{2}{9}$ $(72\frac{7}{9})$
e. 25.6×0.98 (25.088)
f. $4\frac{1}{3} \times 6\frac{2}{7}$ (27.24)
g. $5\frac{2}{3} \div \frac{4}{5}$ $(7\frac{1}{2})$
h. $31 - 5.5$ (25.5)
i. $17\frac{1}{2} \div 2\frac{1}{3}$ $(7\frac{1}{2})$
j. $481.32 \div 0.84$ (573.0)

P411

5.1	7.2	0.3	2.4	4.5
6.9	1.5	2.1	4.2	4.8
1.2	1.8	3.9	6.0	6.6
3.0	3.6	5.7	6.3	0.9
3.3	5.4	7.5	0.6	2.7

8. Copy the square above.

   a. Add the numbers named by the decimals in each column and record the sum for each column. (19.5)

   b. Add the numbers named by the decimals in each row and record the sum for each row. (19.5)

   c. Begin at the lower left-hand corner and add diagonally. Record the sum. (19.5)

   d. Begin in the upper left-hand corner. Add diagonally. Record the sum. (19.5)

   e. Is each sum the same rational number? (yes) What is the number? (19.5)

   f. Is the square a magic square? (yes)

Practice Exercises

I. Find the number that t represents.

a) $5\frac{2}{3} \div 1\frac{1}{9} = t$ $(t = 5\frac{1}{10})$

b) $t \times 64 = 14848$ $(t = 232)$

c) $3\frac{1}{4} \times 2\frac{1}{3} = t$ $(t = 7\frac{7}{12})$

d) $^{-}625 + t = {}^{-}480$ $(t = {}^{+}145)$

e) $7236 \div t = 67$ $(t = 108)$

f) $8\frac{1}{3} = \frac{t}{12}$ $(t = 100)$

g) $^{+}4378 + {}^{-}3487 = t$ $(t = {}^{+}891)$

h) $19 \div \frac{5}{6} = t$ $(t = 22\frac{4}{5})$

i) $^{-}8483 + t = {}^{-}3479$ $(t = {}^{+}5004)$

j) $962.56 \div 6.4 = t$ $(t = 150.4)$

k) $28\frac{9}{16} - t = 12\frac{1}{4}$ $(t = 16\frac{5}{16})$

l) $3\frac{1}{2} \div 1\frac{1}{3} = t$ $(t = 2\frac{5}{8})$

m) $^{+}2740 + t = {}^{+}1769$ $(t = {}^{-}971)$

n) $27\frac{1}{2} \times 5\frac{5}{9} = t$ $(t = 152\frac{7}{9})$

o) $684.84 = 3.9 \times t$ $(t = 175.6)$

p) $7\frac{1}{4} + 19\frac{2}{3} + 11\frac{5}{6} = t$ $(t = 38\frac{3}{4})$

q) $3\frac{4}{7} + t + 3\frac{1}{3} = 12\frac{4}{12}$ $(t = 5\frac{3}{7})$

r) $27.45 + t = 78$ $(t = 50.55)$

s) $\frac{125}{1000} = \frac{t}{8}$ $(t = 1)$

t) $4\frac{5}{6} \times \frac{3}{8} = t$ $(t = 1\frac{13}{16})$

II. Solve

a) $\frac{5}{16} \div 4$ $(\frac{5}{64})$

b) $48.90 \div 30$ (1.63)

c) $5 \times \frac{5}{6}$ $(4\frac{1}{6})$

d) $15.789 + 13.763$ (29.552)

e) $5\frac{1}{3} - 1\frac{3}{7}$ $(3\frac{19}{21})$

f) $\frac{7}{8} \times 36$ $(31\frac{1}{2})$

g) $38.400 \div 60$ $(.640)$

h) $4\frac{1}{2} \div 1\frac{1}{2}$ (3)

i) $38.050 \div 125$ $(.3044)$

j) $8\frac{3}{5} \times 4$ $(34\frac{2}{5})$

k) $428.07 \div .57$ (751)

l) $931.44 - 265.9$ (665.54)

m) $3\frac{1}{12} \times 2\frac{2}{5}$ $(7\frac{2}{5})$

n) $9 \div 4\frac{2}{3}$ $(1\frac{13}{14})$

o) $577.28 \div 6.4$ (90.2)

p) $4\frac{1}{6} \div \frac{3}{4}$ $(5\frac{5}{9})$

q) 162.4×87.5 $(14{,}210)$

r) $468.394 - 288.54$ (179.854)

s) $3\frac{1}{2} + 4\frac{3}{8} + 2\frac{1}{3}$ $(10\frac{5}{24})$

t) $\frac{5}{8} \div 2\frac{1}{2}$ $(\frac{1}{4})$

III. Add:

1.	2.	3.	4.	5.
1463	69053	$2\frac{1}{2}$	843,695	$2\frac{1}{3}$
423	2928	$\frac{5}{6}$	24,763	$6\frac{5}{6}$
4684	75	$4\frac{2}{3}$	927,616	$3\frac{1}{2}$
5736	71089	$1\frac{1}{4}$	44,464	$\frac{3}{4}$
(12,306)	(143,145)	($9\frac{1}{4}$)	(1,840,538)	($13\frac{5}{12}$)

Subtract:

1.	2.	3.	4.	5.
8902	39668	$27\frac{1}{12}$	658.374	$2\frac{4}{5}$
4723	25756	$8\frac{2}{3}$	167.41	$\frac{3}{4}$
(4179)	(13,912)	($18\frac{5}{12}$)	(490.964)	($2\frac{1}{20}$)

Multiply:

1.	2.	3.	4.	5.
365	87.91	$8\frac{5}{8}$	3846	$4.98
427	2.8	$4\frac{2}{3}$	508	36
(155,855)	(246.148)	$40\frac{1}{4}$	(1,953,768)	($179.28)

Divide:

1. $35\overline{)147,535}$ (4215 r10)

2. $24\overline{)5,784}$ (241)

3. $72\overline{)14.544}$ (.202)

4. $4\frac{2}{3} \div 1\frac{1}{6}$ (4)

5. $6 \div \frac{5}{6}$ ($7\frac{1}{5}$)

Braintwister

Egyptian Fractions

The ancient Egyptians expressed all of their fractions as unit fractions, such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{5}$ etc. (One exception was $\frac{2}{3}$) Instead of writing $\frac{7}{12}$ they had to write $\frac{1}{2} + \frac{1}{12}$, or $\frac{1}{4} + \frac{1}{4} + \frac{1}{12}$.

Express each fraction as the sum of two unit fractions.

a) $\frac{9}{16}$ $\left(\frac{1}{2}+\frac{1}{16}\right)$ c) $\frac{5}{8}$ $\left(\frac{1}{2}+\frac{1}{8}\right)$ e) $\frac{1}{3}$ $\left(\frac{1}{4}+\frac{1}{12}\right)$ g) $\frac{3}{8}$ $\left(\frac{1}{4}+\frac{1}{8}\right)$

b) $\frac{5}{12}$ $\left(\frac{1}{3}+\frac{1}{12}\right)$ d) $\frac{5}{6}$ $\left(\frac{1}{2}+\frac{1}{3}\right)$ f) $\frac{3}{4}$ $\left(\frac{1}{2}+\frac{1}{4}\right)$ h) $\frac{9}{20}$ $\left(\frac{1}{5}+\frac{1}{4}\right)$

Review

SET I

Part A

1. Choose an integer from Set A to complete and make each of the following true statements.

Set A = $\{0, {}^{-}1, {}^{-}2, {}^{-}3, {}^{-}4 \ldots\}$

a) $^{+}3 + (\underline{-1}) = {}^{+}2$

b) $(\underline{^{-}7}) + {}^{-}2 = {}^{-}9$

c) $(\underline{-9}) + {}^{+}3 = {}^{-}6$

d) $^{+}10 + (\underline{^{-}13}) = {}^{-}3$

e) $^{+}5 + (\underline{^{-}17}) = {}^{-}12$

f) $(\underline{^{-}30}) + {}^{+}15 = {}^{-}15$

g) $^{+}9 + (\underline{-9}) = 0$

h) $^{+}18 + (\underline{^{-}9}) = {}^{+}9$

2. Choose an integer from Set B to complete and make each of the following true statements.

Set B = $\{0, {}^{+}1, {}^{+}2, {}^{+}3, {}^{+}4 \ldots\}$

a) $^{-}4 + (\underline{^{+}6}) = {}^{+}2$

b) $^{+}5 + (\underline{^{+}11}) = {}^{+}16$

c) $^{+}9 + (\underline{^{+}3}) = {}^{+}12$

d) $(\underline{^{+}8}) + {}^{-}6 = {}^{+}2$

e) $^{-}7 + (\underline{0}) = {}^{-}7$

f) $\underline{+12} + {}^{-}18 = {}^{-}6$

g) $(\underline{^{+}18}) + {}^{-}12 = {}^{+}6$

h) $^{+}3 + (\underline{^{+}48}) = {}^{+}51$

i) The intersection of Sets A and B is Set C. Name the members of Set C. (0)

3. Would the arrow drawn for each of the following unknown addends be named by a positive or negative integer?

a) $^{+}2 + n = {}^{+}8$ (positive)

b) $n + {}^{-}6 = 0$ (positive)

c) $^{+}5 + n = {}^{+}1$ (negative)

d) $n + {}^{+}3 = {}^{-}9$ (negative)

e) $^{+}16 + n = {}^{+}32$ (positive)

f) $n + {}^{+}17 = {}^{-}6$ (negative)

g) $^{-}8 + n = {}^{+}8$ (positive)

h) $^{+}4 + n = {}^{+}3$ (negative)

i) $^{-}15 + n = {}^{+}9$ (positive)

j) $n + {}^{-}1 = {}^{+}21$ (positive)

4. Rename each fraction so that it may be written as a decimal. Complete the chart.

Fraction	Multiply By	New Fraction	Decimal
a) $\frac{1}{5}$	$\frac{2}{2}$ $\frac{1 \times 2}{5 \times 2}$	$\frac{2}{10}$	.2
b) $\frac{4}{25}$	$\left(\frac{4}{4} \quad \frac{4 \times 4}{25 \times 4}\right)$	$\left(\frac{16}{100}\right)$	(.16)
c) $\frac{3}{2}$	$\left(\frac{5}{5} \quad \frac{3 \times 5}{2 \times 5}\right)$	$\left(\frac{15}{10}\right)$	(1.5)
d) $\frac{1}{4}$	$\left(\frac{25}{25} \quad \frac{1 \times 25}{4 \times 25}\right)$	$\left(\frac{25}{100}\right)$	(.25)
e) $\frac{18}{50}$	$\left(\frac{2}{2} \quad \frac{18 \times 2}{50 \times 2}\right)$	$\left(\frac{36}{100}\right)$	(.36)
f) $\frac{7}{20}$	$\left(\frac{5}{5} \quad \frac{7 \times 5}{20 \times 5}\right)$	$\left(\frac{35}{100}\right)$	(.35)
g) $\frac{3}{4}$	$\left(\frac{25}{25} \quad \frac{3 \times 25}{4 \times 25}\right)$	$\frac{75}{100}$	(.75)
h) $\frac{5}{8}$	$\left(\frac{125}{125} \quad \frac{5 \times 125}{8 \times 125}\right)$	$\left(\frac{625}{1000}\right)$	(.625)

5. Find the number that n represents. Use decimal or fraction form for your work. Write the answer as a decimal.

a) .84 × 6.8 = n (n=5.712) e) 1.628 ÷ 4.4 = n (n=.37)
b) 9 ÷ .3 = n (n=30.0) f) .0256 ÷ 1.6 = n (n=.016)
c) 25 × .25 = n (n=6.25) g) .25 × .25 = n (n=.0625)
d) 45.56 ÷ 68 = n (n=.67) h) .25 ÷ 4 = n (n= 0625)

6. Write the set of numbers named by the reciprocals of the members of Set R, call it Set S.

$R = \{1, \frac{1}{2}, \frac{2}{4}, \frac{4}{8}, \frac{6}{12}, \frac{8}{16}, \frac{10}{20}, \frac{12}{24}\}$

$\left(S = \{\frac{1}{1}, \frac{2}{1}, \frac{4}{2}, \frac{8}{4}, \frac{12}{6}, \frac{16}{8}, \frac{20}{10}, \frac{24}{12}\}\right)$

7. Complete the following to make them true sentences.

a) $^{-}23 + \underline{(^{-}18)} = {}^{-}41$

b) $\frac{3}{4} \div \frac{2}{3} = \underline{(1\frac{1}{8})}$

c) $6\frac{2}{3} \times \underline{(8\frac{3}{4})} = 58\frac{1}{3}$

d) $^{-}14 + \underline{(^{+}51)} = {}^{+}37$

e) $1\frac{1}{2} \times \frac{5}{6} = \underline{(1\frac{1}{4})}$

f) $\frac{3}{4} + \underline{(\frac{9}{2})} = \frac{1}{6}$

g) $\frac{2}{3} \times \underline{\frac{5}{4}} = \frac{5}{6}$

h) $\underline{(^{-}24)} + {}^{-}8 = {}^{-}32$

i) $^{+}45 + {}^{-}17 = \underline{(^{+}28)}$

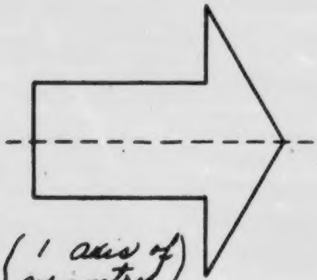
j) $\frac{5}{8} \div 1\frac{1}{4} = \underline{\frac{1}{2}}$

(Answers will vary)

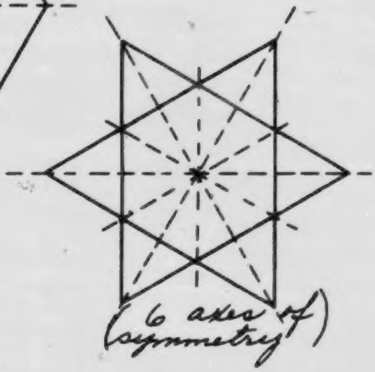
8. Complete the chart below.

Set of Integers	Largest Integer	Smallest Integer
a) $^{-}26, {}^{+}14, {}^{+}26, {}^{-}8, {}^{+}40$	$(^{+}40)$	$(^{-}26)$
b) $^{+}13, {}^{-}9, {}^{+}16, {}^{+}19, {}^{-}8$	$(^{+}19)$	$(^{-}9)$
c) $^{+}4, {}^{-}3, {}^{+}7, 0, +1$	$(^{+}7)$	$(^{-}3)$
d) $^{-}46, {}^{-}28, {}^{+}2, {}^{-}1, {}^{+}279$	$(^{+}279)$	$(^{-}46)$
e) $^{+}7, {}^{+}3, {}^{+}5, 0, {}^{+}2$	$(^{+}7)$	(0)

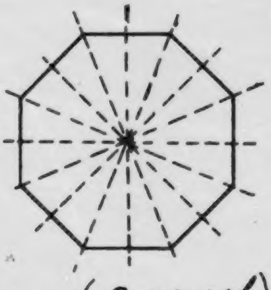
9. Copy these symmetric figures. Draw dotted lines to indicate as many axes of symmetry as you can of each figure.



(1 axes of symmetry)



(6 axes of symmetry)



(8 axes of symmetry)

10. Answer yes or no to these questions.
    - a) Is there a smallest negative integer? (no)
    - b) Is there a largest negative integer? (yes)
    - c) Is there a smallest positive integer? (yes)
    - d) Is there a largest positive integer? (no)
    - e) Is it possible for two fractions to name the same rational number? (yes)
    - f) Is it possible for a decimal to be named by a fraction? (yes)
    - g) Is the set of negative integers closed under addition? (yes)
    - h) Is the set of negative integers closed under subtraction? (no)
    - i) Is the set of positive integers closed under addition? (yes)
    - j) Is the set of positive integers closed under subtraction? (no)
    - k). Is it possible for two fractions to name the same number when they have different denominators? (yes)

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. Mrs. Newlund baked two strawberry pies for dessert. She wants to serve each person $\frac{1}{3}$ of a pie. How many people can she serve with 2 pies? ($2 \div \frac{1}{3} = n$ $n = 6$ She can serve 6 people.)

2. In 1960 the population of Oak Harbor was 2,903. It is estimated that the population will be 2,750 in 1970. What will be the estimated loss in population during these ten years? ($2903 + {}^{-}2750 = n$ $n = {}^{+}153$ The estimated loss will be 153 people.)

3. The Empire State Building is 1,472 ft. high. Without its T.V tower it is 1,250 ft. high. How tall is its T.V. tower? ($^{+}1472 + {}^{-}1250 = n \quad n = 222$
The T.V. tower is 222 feet tall.)

4. Joe arrived at the bus station at 9:32 a.m. His bus is leaving at 10:05 a.m. How long will Joe have to wait for his bus? ($^{+}5 - {}^{-}28 = n \quad n = {}^{+}33$
Joe will have to wait 33 minutes.)

5. During a game of "Ring Toss" Jane's ring fell 24 inches on one side of the peg. Dick's ring fell 19 inches on the opposite side of the peg. How far apart were their rings? ($^{+}24 + {}^{+}19 = n \quad n = {}^{+}43$
Their rings were 43 inches apart.)

6. The water in a swimming pool weighs 148,808 pounds. Water weighs 8.36 pounds each gallon. How many gallons of water are in the pool? ($148{,}808 \div 8.36 = t \quad t = 17{,}800$
The pool has 17,800 gallons of water.)

7. On his route Ralph delivers 84 papers each day and he gets paid $2\frac{1}{4}$ cents for each delivery. How much does he earn in one week? ($(84 \times 2\frac{1}{4}) \times 7 = m \quad m = 13.23$
Ralph earns \$13.23 in one week.)

8. Tractor fuel costs 27.9 cents a gallon and one farmer used 2,430 gallons of fuel this year. How much did he pay for tractor fuel? ($2430 \times 27.9 = c \quad c = 677.97$
The farmer paid \$677.97 for tractor fuel.)

9. In the problem above, the farmer receives a tax rebate of 6 cents a gallon on the fuel. What is the final cost of the tractor fuel? ($2430 \times .06 = t \quad t = 145.80$
$677.97 - 145.80 = c \quad c = 532.17$
The final Cost is \$532.17.)

Review

SET II

Part A

1. Rewrite these sentences using letters to represent the numbers. Examples a and b give you some possible answers.

a) $2 \times 2 = 4$ $a \times a = d$

b) $3 \times 4 = 12$ $a \times b = c$

c) $3 - 0 = 3$ $(a - b = a)$

d) ${}^{+}4 + {}^{-}4 = 0$ $({}^{+}a + {}^{-}a = b)$

e) $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ $\left(\frac{a}{b} + \frac{a}{b} = \frac{c}{b}\right)$

f) $\frac{3}{6} \times \frac{1}{2} = \frac{3}{12}$ $\left(\frac{a}{b} \times \frac{c}{d} = \frac{a}{e}\right)$

g) $6 \div \frac{3}{5} = 3$ $\left(a \div \frac{a}{b} = b\right)$

h) $2 \times 4 \times 9 = 72$ $(a \times b \times c = d)$

i) $12 \div 6 = 2$ $(b \div c = a)$

j) $4.1 + 8.1 = 12.2$ $(a + b = c)$

k) $\frac{9}{12} \div \frac{1}{2} = 1\frac{1}{2}$ $\left(\frac{a}{b} \div \frac{c}{d} = C\frac{c}{d}\right)$

l) $5 \times 5 \times 5 = 5^3$ $(c \times c \times c = c^b)$

2. Write the symbol = or ≠ that makes each of the following a true sentence.

a) $.12 \; \underline{=} \; \frac{3}{25}$

b) $\$259.60 \div 41 \; \underline{\neq} \; \$6.35$

c) $3\frac{2}{7} \div \frac{6}{7} \; \underline{=} \; 3\frac{5}{6}$

d) $5\frac{1}{2} \; \underline{=} \; \frac{44}{8}$

e) $\frac{45}{100} \; \underline{\neq} \; \frac{7}{20}$

f) $20.9 \times 72 \; \underline{\neq} \; 1540.8$

g) ${}^{-}184 + {}^{-}276 \; \underline{=} \; {}^{-}460$

h) $2\frac{1}{4} \; \underline{\neq} \; \frac{225}{1000}$

i) ${}^{-}38 - {}^{+}106 \; \underline{=} \; {}^{-}144$

j) $\frac{3}{8} \; \underline{=} \; .375$

3. Rewrite the following as subtraction sentences and solve. Example a is done for you.

a) ${}^{-}4 + n = {}^{-}5$, $n = {}^{-}5 - {}^{-}4$, $n = {}^{-}1$

b) ${}^{+}6 + n = {}^{-}2$ $({}^{-}2 - {}^{+}6 = {}^{-}8)$

c) $n + {}^{-}9 = {}^{-}12$ $({}^{-}12 - {}^{-}9 = {}^{-}3)$

d) ${}^{+}16 + n = {}^{+}10$ $({}^{+}10 - {}^{+}16 = {}^{-}6)$

e) ${}^{-}8 + n = {}^{+}3$ $({}^{+}3 - {}^{-}8 = {}^{+}11)$

f) $n + {}^{+}21 = {}^{+}7$ $({}^{+}7 - {}^{+}21 = {}^{-}14)$

g) $n + 0 = {}^{-}6$ $({}^{-}6 - 0 = {}^{-}6)$

4. Write the next six members of each set.

a) $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots\}$ $\left\{\frac{4}{8}, \frac{5}{10}, \frac{6}{12}, \frac{7}{14}, \frac{8}{16}, \frac{9}{18}\right\}$

b) $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ $\left\{\frac{3}{1}, \frac{4}{1}, \frac{5}{1}, \frac{6}{1}, \frac{7}{1}, \frac{8}{1}\right\}$

c) $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \ldots\}$ $\left\{\frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \frac{14}{21}, \frac{16}{24}, \frac{18}{27}\right\}$

d) $\{1, \frac{1}{1}, \frac{2}{2}, \ldots\}$ $\left\{\frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \frac{7}{7}, \frac{8}{8}\right\}$

e) $\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \ldots\}$ $\left\{\frac{12}{16}, \frac{15}{20}, \frac{18}{24}, \frac{21}{28}, \frac{24}{32}, \frac{27}{36}\right\}$

5. Tell whether the following statements are true or false.

a) ${}^{+}4 + {}^{-}7 = {}^{-}2 + {}^{-}1$ (T)

b) ${}^{+}3 + {}^{+}6 = {}^{-}6 + {}^{-}3$ (F)

c) $\frac{1}{2} + \frac{3}{4} < 1 + \frac{1}{2}$ (T)

d) $\frac{2}{3} \times \frac{3}{2} > \frac{1}{2} \times \frac{2}{1}$ (F)

e) ${}^{+}5 - {}^{-}2 = {}^{-}5 - {}^{+}2$ (F)

f) $\frac{4}{3} - \frac{2}{3} < \frac{8}{6} - \frac{4}{6}$ (F)

g) ${}^{-}6 + {}^{+}7 = {}^{+}12 + {}^{-}11$ (T)

h) $\frac{3}{6} \times \frac{2}{4} = \frac{1}{4}$ (T)

i) $\frac{3}{16} \div \frac{2}{8} = \frac{9}{6} \quad \frac{6}{3}$ (T)

j) ${}^{-}3 + {}^{-}4 > 0$ (F)

6. Choose another name for the number from the row.

Example: $\frac{1}{10}$	a) 1.	b) $\frac{10}{1}$	[✓ c) .1]	d) .01
$\frac{26}{100}$	a) 2.6	b) $\frac{260}{10}$	c) 26.0	[✓ d) .26]
$3\frac{5}{10}$	a) 0.35	[✓ b) 3.5]	c) $\frac{350}{10}$	d) $\frac{35}{100}$
$12\frac{2}{100}$	a) 12.5	b) $\frac{122}{10}$	[✓ c) 12.02]	d) $\frac{122}{100}$
$\frac{13}{1000}$	a) .130	[✓ b) .013]	c) $\frac{1300}{10}$	d) 1.300

7. Rename each decimal so that it may be written as a fraction. Complete the chart.

Decimal	Fraction Name	Divide by	Fraction (Simnlest form)
a) .5	$\frac{5}{10}$	$\frac{5}{5}$	$\frac{5 \div 5}{10 \div 5}$ $\frac{1}{2}$
b) .15	$(\frac{15}{100})$	$(\frac{5}{5})$	$(\frac{15 \div 5}{100 \div 5})$ $(\frac{3}{20})$
c) .450	$(\frac{450}{1000})$	$(\frac{50}{50})$	$(\frac{450 \div 50}{1000 \div 50})$ $(\frac{9}{20})$
d) .75	$(\frac{75}{100})$	$(\frac{25}{25})$	$(\frac{75 \div 25}{100 \div 25})$ $(\frac{3}{4})$
e) .05	$(\frac{5}{100})$	$(\frac{5}{5})$	$(\frac{5 \div 5}{100 \div 5})$ $(\frac{1}{20})$
f) .08	$(\frac{8}{100})$	$(\frac{4}{4})$	$(\frac{8 \div 4}{100 \div 4})$ $(\frac{2}{25})$
g) .4	$(\frac{4}{10})$	$(\frac{2}{2})$	$\frac{4 \div 2}{10 \div 2}$ $(\frac{2}{5})$
h) .125	$(\frac{125}{1000})$	$(\frac{125}{125})$	$(\frac{125 \div 125}{1000 \div 125})$ $(\frac{1}{8})$

8. Each of the following is true by one of the properties of rational numbers. Use the first letter of each word to identify the property. Example: D P A for distributive property of addition.

a) $\frac{3}{4} \times \frac{5}{6} = \frac{5}{6} \times \frac{3}{4}$ (C.P.M.)

b) $\frac{1}{2} + (\frac{1}{4} + \frac{1}{5}) = (\frac{1}{2} + \frac{1}{4}) + \frac{1}{5}$ (A.P.A.)

c) $3 \times 2\frac{1}{2} = (3 \times 2) + (3 \times \frac{1}{2})$ (D.P.M.)

d) $\frac{1}{3} \times (3 \times 6) = (\frac{1}{3} \times 3) \times 6$ (A.P.A.)

e) $4\frac{3}{4} \times 4\frac{1}{2} = (4 \times 4) + (\frac{3}{4} \times \frac{1}{2})$ (D.P.M.)

f) $\frac{4}{5} + \frac{2}{3} = \frac{2}{3} + \frac{4}{5}$ (C.P.A.)

g) $1 \times \frac{7}{8} = \frac{7}{8}$ (Identity)

h) $8\frac{2}{3} \times \frac{1}{4} = (8 \times \frac{1}{4}) + (\frac{2}{3} \times \frac{1}{4})$ (D.P.M.)

i) $\frac{9}{10} \times 1 = \frac{9}{10}$ (Identity)

j) $.75 \times 2.5 = 2.5 \times .75$ (C.P.M.)

P422

9. Perform the operations in Column 1 and match with the correct result in Column 2.

	Column I	Column II
a)	$^{-}3{,}682 + {}^{+}1{,}463$	(g) $2\frac{4}{15}$
b)	$6\frac{1}{5} \div 1\frac{2}{3}$	(f) $\frac{20}{29}$
c)	$2\frac{1}{4} \times \frac{7}{8}$	(e) $44\frac{1}{24}$
d)	$\frac{136}{32}$	(b) $3\frac{18}{25}$
e)	$17\frac{3}{8} + 26\frac{2}{3}$	(c) $1\frac{31}{32}$
f)	$2\frac{1}{2} + 3\frac{5}{8}$	(a) $^{-}2{,}219$
g)	$5\frac{3}{5} - 3\frac{1}{3}$	(d) 4.25
h)	$\frac{127}{25}$	(h) 5.08

10. Complete these statements with the word needed to make them true.

a) The product of two rational numbers is always a (rational) number.

b) A fraction with a denominator of 10 , 100 , 1,000 , ... may be written as a (decimal) .

c) To write the reciprocal of a fraction, we (invert) the fraction.

d) The product of a fraction and its (reciprocal) is 1.

e) One divided by a fraction is the (reciprocal) of the fraction.

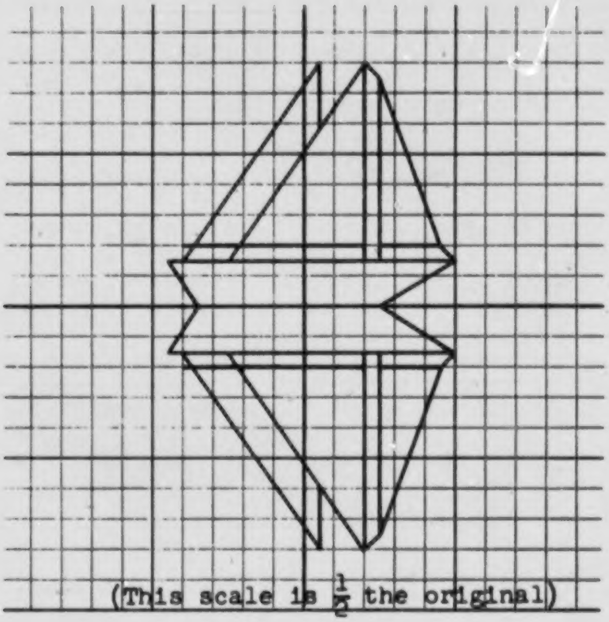
f. To divide by a fraction (multiply) by its reciprocal.

11. Graph the following ordered pairs:

A	$(^{+}5 , 0)$	G	$(^{+}5 , ^{+}3)$	M	$(^{-}8 , ^{+}3)$
B	$(^{+}10 , ^{+}3)$	H	$(^{+}4 , ^{+}3)$	N	$(^{-}7 , 0)$
C	$(^{+}9 , ^{+}4)$	I	$(^{+}4 , ^{+}4)$	P	$(^{-}9 , ^{+}3)$
D	$(^{+}5 , ^{+}15)$	J	$(^{+}1 , ^{+}11)$	Q	$(^{-}5 , ^{+}3)$
E	$(^{+}4 , ^{+}16)$	K	$(^{+}1 , ^{+}16)$	R	$(^{+}1 , ^{+}3)$
F	$(^{+}5 , ^{+}4)$	L	$(^{-}8 , ^{+}4)$	S	$(^{+}1 , ^{+}4)$

Draw $\overline{AB}$, $\overline{QB}$, $\overline{PN}$, $\overline{CB}$, $\overline{CD}$, $\overline{DE}$, $\overline{DG}$, $\overline{FC}$, $\overline{EH}$, $\overline{IL}$, $\overline{EQ}$, $\overline{KM}$, $\overline{KJ}$, and $\overline{RS}$.

Graph the reflection on the x-axis and you have a calm day scene.



(This scale is $\frac{1}{2}$ the original)

P424

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. The speed of sound is generally 1088 feet per second. What is the speed of sound in miles per hour? [$(1088 \times 60 \times 60) \div 5280 = n$ or $1088 \times 60 \times 60 = 3,916,800$; $3,916,800 \div 5280 = n$, $n = 741.8$
the speed of sound is generally 741.8 m.p.h..]

2. A plane flying at the speed of sound has a speed of "Mach I". How far will a plane flying at "Mach I" travel in thirty minutes? ($741.8 \div 2 = n$, $n = 370.9$
The plane will travel 370.9 miles)

3. A fisherman cast his bait 120 ft. upstream. He reeled it in after it had floated 75 ft. downstream. How far did he have to reel it in? ($^{+}120 + {}^{-}75 = n$, $n = {}^{+}45$
He had to reel it in 45 ft..)

4. On Miss Meek's map 1 inch represented 12 miles. The distance from St. Louis to Bellville was $3\frac{3}{4}$ inches. What is the distance in miles from St. Louis to Bellville?
($3\frac{3}{4} \times 12 = n$, $n = 45$ The distance is 45 miles.)

5. Mary went into business. The first month she lost $400.00. The second month she made a profit of $273.00. At the end of the second month has Mary made a loss or a profit? How much? ($^{-}400 + {}^{+}273 = n$, $n = {}^{-}127$ Mary has a loss of $127.00)

6. Octavius, an ancient Roman, was born in 75 B.C. He died in 37 B.C. How old was he when he died? ($^{-}75 + {}^{+}37 = n$, $n = {}^{-}38$
Octavius was 38 years old)

7. If you can cut twelve slices from one watermelon, how many slices can you cut from $2\frac{1}{4}$ watermelons? ($12 \times 2\frac{1}{4} = n$ $n = 27$
you can cut 27 slices.)

8. The earth weight of a moon probe is 822 pounds. The moon weight is one-sixth that of the earth. What is the weight of the probe on the moon? ($822 \times \frac{1}{6} = w$, $w = 137$ The probe weighs 137 lbs. on the moon.)

Review

SET III

Part A

1. Which of these are names for $^{+}3$? $(^{+}8 - ^{+}5,\ ^{+}5 + ^{-}2,\ ^{-}1 + ^{+}4)$

$^{+}5 + ^{-}2$, $^{-}1 + ^{-}2$, $^{+}8 - ^{+}5$, $^{-}1 + ^{+}4$

Which of these are names for $^{-}1$? $(^{+}45 + ^{-}46,\ ^{-}146 + ^{+}145)$

$^{+}7 + ^{+}6$, $^{+}45 + ^{-}46$, $^{-}146 + ^{+}145$, $^{+}27 + ^{-}26$

Which of these are names for $^{-}82$? $(^{-}1462 + ^{+}1380,\ ^{+}869 + ^{-}951)$

$^{-}1462 + ^{+}1380$, $^{+}749 + ^{-}832$, $^{+}869 + ^{-}951$

Which of these are names for 0 ? $(^{+}42 + ^{-}42,\ ^{-}83 + ^{+}83)$

$^{+}8 + 0$, $^{+}42 + ^{-}42$, $^{-}16 + 0$, $^{-}83 + ^{+}83$

2. Using the symbol <, =, or > make each of the following a true sentence.

a) $^{-}27 + ^{+}8 \ \underline{(<)}\ ^{+}27 + ^{-}8$

b) $^{+}293 + ^{-}314 \ \underline{(=)}\ ^{+}319 + ^{-}340$

c) $^{+}37 + 0 \ \underline{(>)}\ ^{+}216 + ^{-}216$

d) $^{-}571 + ^{+}589 \ \underline{(=)}\ ^{-}246 + ^{+}264$

e) $^{+}724 + ^{-}837 \ \underline{(<)}\ ^{+}837 + ^{-}724$

f) $^{+}764 + ^{-}961 \ \underline{(=)}\ ^{-}103 + ^{-}94$

g) $^{+}26 + ^{+}85 \ \underline{(>)}\ ^{+}193 + ^{-}294$

h) $^{-}283 + ^{+}13 \ \underline{(>)}\ ^{-}271 + 0$

i) $^{-}47 + ^{-}68 \ \underline{(<)}\ ^{+}808 + ^{-}115$

j) $^{+}709 + ^{-}698 \ \underline{(=)}\ ^{-}698 + ^{+}709$

3. R {The set of rational numbers}

In which subset of Set R will the number that n represents be?

Example a is done for you.

a) $\frac{1}{2} + n = \frac{3}{4}$ {fractions}

b) $.02 \times 6.1 = n$ {decimals}

c) $6 \times n = 0$ {whole}

d) $^{+}5 + n = {}^{+}2$ {integers}

e) $342 - 342 = n$ {whole}

f) $16 \div \frac{3}{4} = n$ {fractions}

g) $4 - 6 = n$ {integers}

h) $1 + n = 2$ {whole or Counting}

4. Solve the following both as decimals and as fractions. An example is shown.

Example: $\frac{3}{4} \div \frac{1}{2}$, $.75 \div .5 = 1.5$, $\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{6}{4} = 1\frac{1}{2}$

a) $\frac{1}{5} + \frac{1}{2}$ (.7 or $\frac{7}{10}$)

b) $3\frac{1}{4} - \frac{3}{4}$ (2.5 or $2\frac{1}{2}$)

c) $.15 \times 1.4$ (.210 or .$\frac{21}{100}$)

d) $7.2 \div 9$ (.8 or $\frac{4}{5}$)

e) $.4 + 2.5 + .7$ (3.6 or $3\frac{3}{5}$)

f) $7.25 \div .25$ (29)

g) $2\frac{2}{5} \times \frac{3}{10}$ (.72 or $\frac{18}{25}$)

h) $3.6 - .35$ (3.25 or $3\frac{1}{4}$)

i) $7\frac{1}{2} \div \frac{3}{10}$ (25)

j) $5\frac{4}{5} \times 1\frac{1}{2}$ (8.70 or $8\frac{7}{10}$)

5. Use the words closed or not closed to complete and make these true sentences.

a) The set of rational numbers is (Closed) under the operation of multiplication.

b) The set of rational numbers greater than zero is (not Closed) under the operation of subtraction.

c) The set of rational numbers is (Closed) under the operation of addition.

d) The set of rational numbers greater than zero is (not Closed) under the operation of division.

6. Find n in each of the following.

a) $4 \times n = {}^{-}28$ $(n = {}^{-}7)$

b) $n + 1\frac{4}{7} = 3\frac{9}{14}$ $(n = 2\frac{1}{14})$

c) ${}^{+}168 + n = {}^{-}312$ $(n = {}^{-}480)$

d) $\frac{5}{8} + 3\frac{1}{3} = n$ $(n = \frac{3}{16})$

e) $17.4 \times n = 400.2$ $(n = 23)$

f) ${}^{+}94 + {}^{-}36 = n + {}^{-}214$ $(n = {}^{+}272)$

g) $129.92 \times n = 44.8 \times 2.9$ $(n = 1)$

h) $1\frac{3}{4} + n = 6\frac{2}{3}$ $(n = 4\frac{11}{12})$

i) $27.38 + 12.62 = 40 + n$ $(n = 0)$

j) $5\frac{1}{2} \times 2\frac{1}{3} = n + 6\frac{1}{2}$ $(n = 6\frac{1}{3})$

7. Graph this set of points $S = \{A({}^{-}4, {}^{+}4), B({}^{+}6, {}^{+}4)$ $C({}^{+}2, {}^{+}4), D({}^{-}2, {}^{+}4) \ldots\}$ Draw $\overline{AD}$. Extend $\overline{AD}$. The points of line AD seem equidistant to all points on which axis? (x axis)

$T = \{F({}^{-}3, {}^{+}6), G({}^{-}3, {}^{-}6), H({}^{-}3, {}^{+}2), J({}^{-}3, {}^{-}1) \ldots\}$

Graph Set T. Draw $\overline{FG}$. Extend $\overline{FG}$. The points of line FG seem equidistant to all points of which axis? (y axis)

R = {The set of points with y-coordinate 8}

W = {The set of points with x-coordinate 6}

Which set of points is equidistant to the y-axis? x-axis?
(Points of set R are equidistant from x axis. The points of set W are equidistant from y axis)

8. Which of the following have fraction names with the same numerator? [(a, d, h), (c, f, j), (b, e, g, i)]

a) 6.23

b) 45.

c) 1.7

d) 623

e) 0.45

f) .17

g) .045

h) 62.3

i) 4.5

j) 17

9. Which of the numbers above have fraction names with the same denominator? [(a, e, f), (c, h, i), (b, d, j)

10. Are any two of the numerals above names for the same number? (no)

11. Use the correct sign of operation to make true number sentences. Example a is worked.

a) $54 \; \underline{\;\div\;} \; 6 \; \underline{\;+\;} \; 8 \; \underline{\;=\;} \; 17$

b) $2\frac{1}{2} \; \underline{(-)} \; 1\frac{1}{4} \; \underline{(+)} \; 1\frac{1}{4} \; \underline{(=)} \; 2\frac{1}{2}$

c) $^{+}19 \; \underline{(-)} \; ^{+}14 \; \underline{(=)} \; ^{-}28 \; \underline{(+)} \; ^{+}33$

d) $7 \; \underline{(x)} \; 4 \; \underline{(=)} \; 2 \; \underline{(x)} \; 14$

e) $\frac{2}{3} \; \underline{(x)} \; \frac{1}{2} \; \underline{(=)} \; \frac{1}{3} \; \underline{(x)} \; \frac{1}{1}$

f) $^{-}22 \; \underline{(+)} \; ^{+}16 \; \underline{(-)} \; ^{-}6 \; \underline{(=)} \; 0$

g) $\frac{3}{4} \; \underline{(\div)} \; \frac{1}{3} \; \underline{(x)} \; \frac{4}{2} \; \underline{(=)} \; \frac{9}{2}$

h) $\frac{5}{8} \; \underline{(-)} \; \frac{1}{4} \; \underline{(+)} \; \frac{3}{4} \; \underline{(=)} \; \frac{9}{8}$

12. What would be the decimal name of the last place to the right needed in:

Example a) The product of tens and tenths? ones

b) The product of tenths and tenths? (hundredths)

c) The product of hundreds and tenths? (tens)

d) The product of hundreds and hundredths? (ones)

e) The product of hundredths and tenths? (thousandths)

f) The product of thousands and thousandths? (ones)

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. Mr. Hank's car went 244.8 miles on 17 gallons of gasoline. At this rate, how far would he go on 12.5 gallons? ($244.8 \div 17 = a$ $a = 14.4$ $14.4 \times 12.5 = d$ $d = 180$ He would go 180 miles.)

2. While baby-sitting for a neighbor, Marge receives 45 cents an hour. Last year she was paid $32.40. How many hours did she spend baby-sitting? ($32.40 \div .45 = t$ $t = 72$ Marge spent 72 hours baby-sitting.)

3. The area of the state of Oregon is 96,981 square miles. West Virginia is about $\frac{1}{4}$ the size of Oregon. What is the approximate area of W[illegible] Virginia? ($96,981 \times \frac{1}{4} = a$ or $96,981 \div 4 = a$ $a = 24,2$[illegible] West Virginia is about $24,245\frac{1}{4}$ sq. mi. in area.)

4. The area of West Virginia was found to be 24,181 square miles. How much too large was the approximate area found in Problem 3 above? ($24,245\frac{1}{4} - 24,181 = c$ $c = 64\frac{1}{4}$ The approximate area was $64\frac{1}{4}$ sq. mi. too large.)

5. A farmer used 66.5 bushels of bean seed. It takes 1.75 bushels of seed to plant an acre. How many acres of beans did he plant? ($66.5 \div 1.75 = a$ $a = 38$ He planted 38 acres of beans.)

6. The Taylor Oil Co. drilled an oil well. They struck oil at a depth of 5728 feet. The oil gushed from the well to a height of 169 feet. How far did the oil "gush" altogether? (${}^{-}5728 - {}^{+}169 = n$ $n = {}^{-}5897$ The oil gushed 5,897 feet altogether.)

7. The scale of miles on one map uses one inch to represent 160 miles. The air miles between two cities is 3,000 miles. What is the scale measurement in inches? ($3000 \div 160 = d$ $d = 18\frac{3}{4}$ The scale measurement is $18\frac{3}{4}$ inches.)

8. John can walk 2 miles in 25 minutes. One morning he walked $\frac{3}{5}$ of this distance in $\frac{1}{2}$ the time. How far did he walk and how long did it take him? ($\frac{3}{5} \times 2 = n$ $n = 1\frac{1}{5}$ $\frac{1}{2} \times 25 = t$ $t = 12\frac{1}{2}$ John walked $1\frac{1}{5}$ miles in $12\frac{1}{2}$ minutes.)

Individual Projects

1. Obtain a micrometer and learn to use it. A micrometer is an instrument that can make a measurement to the nearest thousandth of an inch. Demonstrate its use to your class.

2. Scientific notation is a name given to a quick and easy way to represent either very large or very small numbers. Find an explanation of this notation. Write some very large or very small numbers in scientific notation for your class.

3. Many great men have made important contributions to mathematics. Make a report about one of these famous mathematicians and his contribution.

4. There are some mathematical problems that are still mysteries to mathematicians. There are some about prime numbers, odd numbers and construction using only compass and straightedge. Look up one of these problems and try to solve it.

Group Activity

"Travel"

The object of the game is to "travel" as far as possible by being faster with the correct response. The first player stands beside the desk of the next player. A problem is given orally or a card flashed. The child giving the correct response first moves on to stand beside the next player's desk. For each turn the loser remains at the seat where he lost.

Chapter 7

VOLUME

PURPOSE OF UNIT

This is the fourth of a series of Units which comprise an introduction to the study of measurement in the elementary grades. Linear Measurement in the fourth grade, Measurement of Angles and Area in the fifth grade, and Volume in the sixth grade provide the child with a first experience in a branch of geometry which is a powerful tool in the physical world. The four Units present a continuous and coordinated development of measurement.

The Unit on Volume assumes a background of successful completion of the other three units. Consequently, the teacher and the students should be familiar with both the underlying philosophy and the language of the Units referred to above. In those Units and also in this one on Volume, the sequence of topics follows a general pattern:

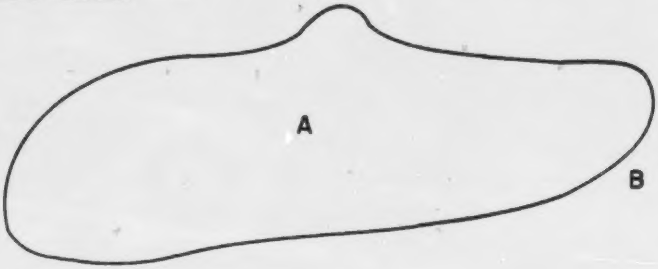
(1) intuitive awareness of comparisons of size,

(2) selection of a particular space region as a unit of volume,

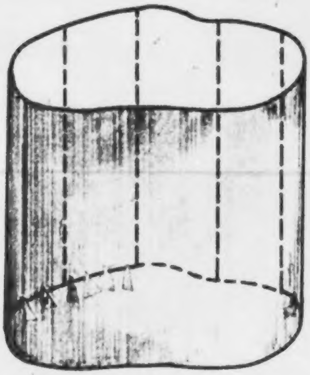
(3) measurement of volumes in terms of this unit, and

(4) selection of standard units of volume for purposes of effective communication. The study of Volume also builds upon the student's ability to recognize the more common space figures.

Every effort has been made to be consistent in the language throughout the four units. A plane figure is any set of points in a plane. Correspondingly, we call a solid figure any set of points in space, not all in the same plane. A simple closed curve like



does not cross itself, and divides the plane in which it lies into 2 sets of points (a) the set of points, A, interior to the curve, and (b) the set of points, B, exterior to the curve. The union of the simple closed curve and its interior is called a plane region. Similarly, a simple closed surface like



does not cross itself and divides space into 3 sets of points, (a) the set of points in the interior of the surface, (b) the set of points of the surface, and (c) the set of points exterior to the surface. Just as it is not possible to pass from the interior of a simple closed curve to the exterior without crossing the curve, it is also not possible to pass from the interior of a simple closed surface to the exterior without crossing the surface. The union of a simple closed surface and its interior is called a space region.

We used a line segment to which we assigned a measure of 1 as a unit for determining the length of a given line segment. We used a plane region to which we assigned a measure of 1 as a unit for determining the area of a given plane region. Now we use a solid region to which we assign a measure of 1 as a unit for determining the volume of a given solid region.

Object to be Measured	Unit to be used to find the measure
1. line segment	1. line segment whose length is 1 inch
2. plane region	2. plane region bounded by a square whose length and width are each 1 inch, making the area of the plane region 1 square inch
3. solid region	3. solid region bounded by a cube whose length, width, and height are each 1 inch, making the volume of the solid region 1 cubic inch

In measurement, only the length of a line segment concerns us. For a plane region we are interested in its area and also in the length of its bounding curve, (perimeter). For a solid region we may want to know, in addition to its volume, the area of the simple closed surface bounding the solid region and also if parts of the bounding surface are plane regions, the perimeters of the simple closed curves bounding these regions. However, when we speak of the measure of a line segment, we mean the number associated with its length; when we speak of the measure of a solid region we mean the number associated with the volume. We continue to stress that the calculations are performed with the measures, i.e., the numbers. We do not have the algebraic apparatus to multiply inches by inches by inches. The student must not be left with the impression that 4 inches × 2 inches × 5 inches equals (4 × 2 × 5) × (inches × inches × inches) equals 40 cubic inches. If the length, width, and height of a box are 4 inches, 2 inches, and 5 inches respectively, then the measure of the box is 4 × 2 × 5 or 40. This means that 40 unit solid regions, each with a volume of 1 cubic inch, would fill the interior of the box without overlapping. Hence, we say the volume of the box is 40 cubic inches.

Fundamentally, the pedagogical approach to this unit on Volume, like the approach of the three other units on measurement, is intuitive. The following basic assumptions, tacitly assumed though not verbalized in the student text, parallel similar assumptions for linear measurement, angle measurement, and area:

a. Two solid regions whose bounding surfaces are congruent have the same volume.

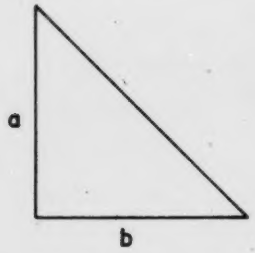
b. If we fill the interior of a solid region with other solid regions that do not overlap, then the sum of the measures of all the solid regions in the interior of the given solid region is the same as the measure of the given solid region.

In addition to assuming familiarity with the skills and understandings of the geometry units studied in the 4th and 5th grades, the unit on Volume also builds on those fundamental concepts of the structure of arithmetic which were previously presented to the pupils. The properties of commutativity and associativity for multiplication (re-ordering and re-grouping the factors of a product) are used in considering the change in the volume of a rectangular prism if the number of units in the width and height are doubled or halved. The parallel geometric development of this study in variation in the text offers a prime opportunity to stress the basic philosophy of teaching an integrated course in mathematics at all levels, rather than departmentalized and unrelated bodies of knowledge entitled "arithmetic," "algebra," "geometry," etc.

This unit provides many experiences in arriving at a reasonable guess as a result of ordinary induction (taking several examples and studying the results for some general pattern). It also introduces the pupil to a technique for recording a deductive argument in mathematics. Heretofore, any deductive arguments expected of a pupil were of a most informal nature. Now the pupil has presented to him a set of related statements leading to a conclusion, and beside each statement is the reason for which he thinks his statement is true. The teacher will need to use his good judgment in determining the amount of stress to be placed on this formal presentation of a proof with his particular class.

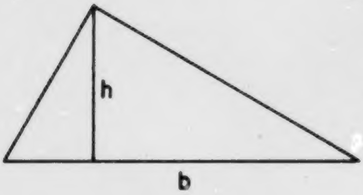
In the interests of efficiency various teaching techniques are suggested for this unit. Where the gathering of materials is more challenging, as in the immersion experiments, it might be wisest for the lesson to be a teacher demonstration with the pupils participating in formulating the conclusions. Where the handling of the materials, like flour, rice, sugar, water, etc., by more than one pupil at a time might be messy, the lesson could well consist of a series of experiments presented by individual pupils. In general, the format of the unit follows that of the three previous units on measurement: exploration as a teacher-directed group activity, and then opportunity for independent study. The exercises do not serve merely as drill. Frequently, important conclusions are meant to be reached by the pupils as a result of their independent efforts with the exercises. Often questions posed to the pupils in the exploration section are not answered in the student text, in order not to stifle any mental activity. Answers in the teacher's commentary may be helpful to the teacher in guiding the students towards discovery.

__Formulas__ __used__ __in__ __this__ __unit__: (Note: "$A_{\text{right triangle}}$" is read "The number of square units in the area of the plane region bounded by a right triangle," and "$V_{\text{rectangular prism}}$" is read "The number of cubic units in the volume of the solid region bounded by a rectangular prism.")



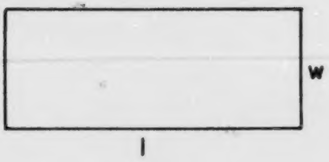
$A_{\text{right triangle}}$ = one-half the product of the measures of the two sides on the right angle

$= \frac{1}{2} \times (a \times b)$



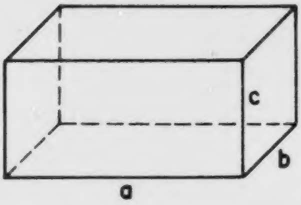
A_{triangle} = one-half the product of the measure of the base and the measure of the altitude to that base

$= \frac{1}{2} \times (b \times h)$



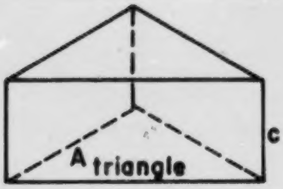
$A_{\text{rectangle}}$ = product of the number of units in the length and in the width

$= \ell \times w$



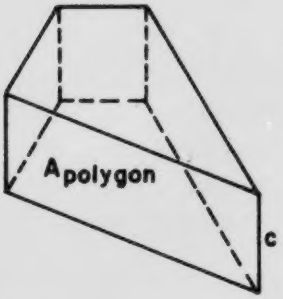
$V_{\text{rectangular prism}}$ = product of the number of units in the length, the width, and the height

$= a \times b \times c$



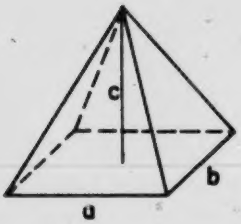
$V_{\text{triangular prism}}$ = product of the measure of the base and the number of units in the height

= $A_{\text{triangle}} \times c$



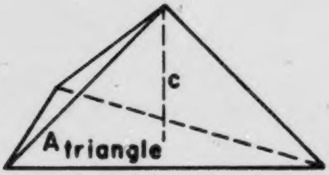
V_{prism} = the product of the measure of the base and the number of units in the height

= $A_{\text{polygon}} \times c$



$V_{\text{rectangular pyramid}}$ = one-third the product of the number of units in the length, in the width, and in the height

= $\frac{1}{3} \times (a \times b \times c)$



$V_{\text{triangular pyramid}}$ = one-third the product of the measure of the base and the number of units in the height

= $\frac{1}{3} \times (A_{\text{triangle}} \times c)$

Understanding and skills to be obtained.

1. Some things occupy more space than others. Frequently we can tell just by looking at two objects which is larger (i.e. occupies more space).
2. The shape of two objects may be entirely different and they will still occupy the same amount of space.
3. Weight is not an indication of space occupied. A three-pound bag of chicken feathers occupies more space than a three-pound block of lead.
4. We may compare the sizes of objects by pouring sugar or sand or rice or water, etc., into the interiors and observing which holds more. This procedure at best gives only a gross comparison, and can be used only for relatively small objects.
5. We need some mathematical procedure for measuring the space occupied by an object. Comparison of sizes of objects would then be much simpler.
6. Any set of points in space is called a space figure.
7. A simple closed surface is a space figure which divides space into two parts, an interior and an exterior set.
8. A space region is the union of a simple closed surface and its interior.
9. The measure of a line segment is the number of units in its length.
10. The measure of a simple closed surface is the number of square units in its area.
11. The measure of a space region is the number of cubic units in its volume.
12. The volume of a space region tells the measure of the region and the name of the unit used to determine the amount of space.

13. A space region bounded by a cube is a more efficient unit of measure than a space region bounded by a sphere or a cylinder.

14. A space region bounded by a cube whose edge is 1 inch long is the standard unit for finding the volume of a given space region. We call this unit the cubic inch.

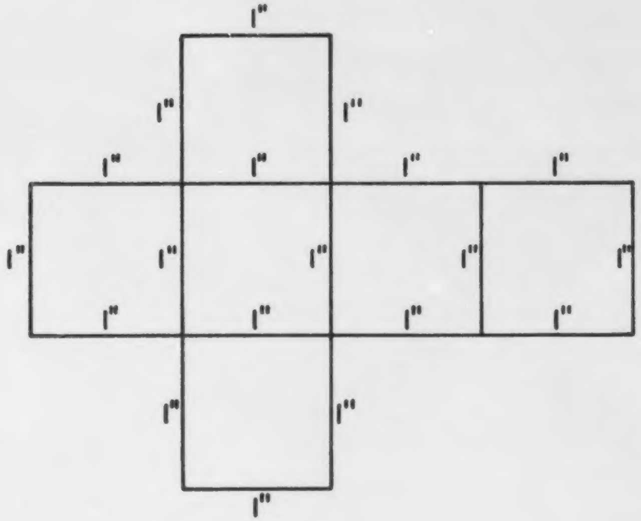
15. The space region bounded by a cube whose edge is 1 foot or 1 yard is also used as a unit of measure of a space region. We call these units the cubic foot and the cubic yard.

16. It would take 1728 cubic inches to fill the interior of a cubic foot.

17. It would take 27 cubic feet to fill the interior of a cubic yard.

Materials

Teacher: It would be helpful to have any easily available models of rectangular prisms: chalk box, milk carton, wood block, rectangular baking tin, shoe box, large grocery carton. A milk carton could be cut and rebuilt to have a model of a triangular prism. Large cylindrical can, paper, scotch tape, scissors, pattern to make a cubic inch.



Pupils: Depends upon how much constructing you want the pupils to do.

Vocabulary: Volume, space figure, simple closed surface, space region, rectangular prism, cube, triangular prism, congruent, cubic inch, cubic foot, cubic yard.

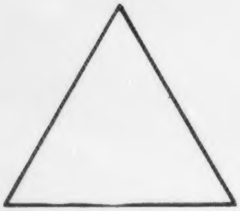
Chapter 7

VOLUME

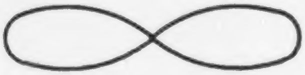
SPACE FIGURES AND SPACE REGIONS

Any set of points in a plane is called a plane figure. Thinking in the same way, we shall call any set of points in space a space figure. The set of all points on a sphere is an example of a space figure. The surface of a box is a model of another space figure.

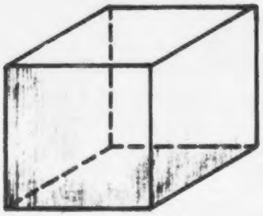
Which of the following are pictures of plane figures? Which ones are pictures of space figures?



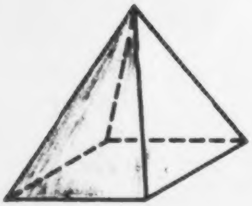
A.
(plane figure)



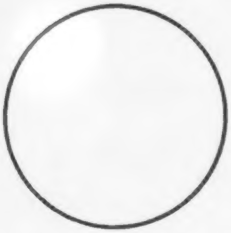
B.
(plane figure)



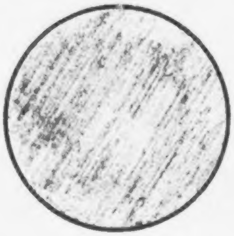
C.
(space figure)



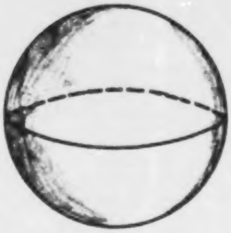
D.
(space figure)



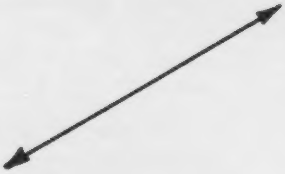
E.
(plane figure)



F.
(plane figure, disk)

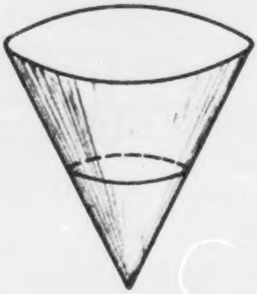


G.
(space figure, sphere)

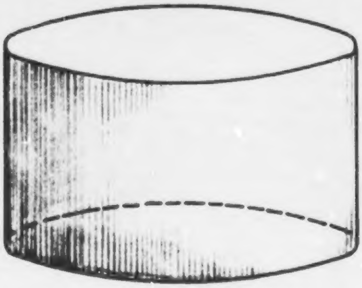


H.
(plane figure)

I.
(plane figure, 3 non-collinear points)



J.
(space figure, cone)



K.
(space figure, cylinder)

Every (plane) figure is also a (space) figure.

The set of all points on a circle is a space figure. So is the set of all points in a circular region. These space figures are also plane figures. The peel covering an orange is a model of a space figure. An ice cream cone is another model. These last two space figures are not plane figures.

1. a. You will remember that simple closed curves are special plane figures. They are those parts which do not cross themselves and which divide the plane in which they lie into three parts: the curve itself, the part called the interior of the curve, and the part called the exterior of the curve.

   b. Which figures shown on the previous page are pictures of simple closed curves? Color the interior of each of these.

2. a. A simple closed surface is a space figure which does not intersect itself and which divides the space in which it lies into three parts: the simple closed surface, the part called the interior of the surface, and the part called the exterior of the surface. The peel covering an orange is a model of a simple closed surface. The part of the orange that you eat is a model of the interior of this surface. All the points of space not covered by the orange make up the exterior.

   Which of the figures on the previous page are pictures of simple closed surfaces? Describe the interior of each of these.

   b. The union of a simple closed surface and its interior is called a space region. The whole orange is a model of a space region.

   Which of the figures are pictures of space regions?

Exercise Set 1

Decide which of the following represent space regions, which represent the interior of a space region, and which represent simple closed surfaces.

1. A walnut, (space region) the walnut shell, (simple closed surface) the walnut meat. (interior of a space region)

2. A can of peas, (space region) the can, (simple closed surface) the contents of the can (interior of a space region)

3. A bottle of soda, (space region) the bottle itself with cap. (simple closed surface)

4. The walls, floor, and ceiling of a room. (simple closed surface)

5. A block of ice. (space region)

6. A jar full of water, (space region) the water in the jar, (interior of a space region) the jar and lid. (simple closed surface)

7. A hollow rubber ball, (simple closed surface) the rubber ball and the air inside the ball. (space region)

8. An empty shoe box plus the set of points in its interior. (space region)

P435

COMPARISON OF SIZES OF OBJECTS

Exploration

Robert bought his father a set of hair brushes for Father's Day. The lady in charge of gift wrapping had to try several boxes before she found one into which the brushes fit properly. How could she decide if the box was too small? too large?

Can your mother fit a 30 pound turkey into the oven in your kitchen?

Do all your text books fit into your brief case?

Is an overnight bag large enough to carry all the clothing you will need for a month's vacation?

Do you think five bus loads of children could fit into one school bus?

In every question raised above, the space occupied by some object was of interest to us. In each case, tell why this was so. We also needed to know when one object was larger than another.

You have already had some practice comparing the sizes of line segments and plane regions. When you have two models of line segments (or plane regions) you have a way of telling whether they are equal. If they are not equal, you can usually decide which one is larger. Can you make this kind of a decision about two space regions? Try the problems in Exercise Set 2.

Exercise Set 2

In each of the following exercises, decide which of the two models represents the larger space region. For each model you should be able to tell which part of the model represents the interior and surface.

1. Your classroom or the school auditorium. (school auditorium)

2. A quart jar of milk or a gallon jar of water. (gallon jar of water)

3. An orange or a grapefruit. (grapefruit, usually)

4. A family car or a bus. (bus)

5. A covered two-quart saucepan or a covered two-quart frying pan. (the measure of the space region should be the same, even though different shapes are involved.)

6. A tennis ball or a golf ball. (tennis ball)

Did you know the answer to each of the above exercises at once? Were there some questions for which you were not sure of the answer? How did you decide?

Suppose that we have a pair of shoe boxes. One way in which we could compare two such boxes would be to compare the amounts of contact paper needed to cover the outside of these boxes.

If we make this sort of comparison, what property of the boxes are we interested in? (surface area)

We might, however, want to compare the amounts that the boxes could hold. Would we need to know the areas of the surfaces of these boxes before we could make this comparison?

Would it help to know the size of the space region enclosed by this box?

Exercise Set 3

In each case tell whether you are interested in the area of the simple closed surface or the size of the space region it encloses.

1. How much paint will you need to cover the inside of a toy chest. (area of the simple closed surface)

2. Which of two toy chests will provide more storage space. (the size of the space region it encloses)

3. Which of two toys will fit in a gift box. (the size of the space region it encloses)

4. How much material you will need to recover a doll pillow. (the area of the simple closed surface)

5. How many marbles you can put in a box. (the size of the space region it encloses)

6. Make two other examples. Make one of them an example in which you would be interested in the area of a simple closed surface and one in which you would be interested in the size of a space region. (Answers will vary.)

COMPARING SPACE REGIONS

Exploration

Two space regions may be compared by seeing whether one may be included in the other. If one space region can be placed entirely in the interior of the other, the first space region is said to be smaller than the second. This would help us decide that a classroom is smaller than the school auditorium and that a marble is smaller than a beach ball.

We could also decide in this way that an orange is smaller than a grapefruit. Comparing the size of these two pieces of fruit by actually seeing whether one may be included in the other is a little harder. But the orange and the grapefruit are only models of space regions and not the regions themselves. If, without doing too much damage to the peel, we were to remove the parts of the grapefruit that we eat, it is clear that the orange would fit inside the grapefruit.

Exercise Set 4

For each of the following exercises decide which of the two models represents the smaller space region.

1. A baseball or a basketball (baseball)

2. A shoe box or a hat box. (cannot tell)

3. A milk bottle or a pop bottle. (cannot tell)

4. A pear or a banana. (cannot tell)

5. A candy bar or an apple. (cannot tell)

6. A juice glass full with orange juice or an empty cup. (cannot tell)

7. An empty pencil box or a full waste paper basket. (an empty pencil box)

8. A pitcher full with water or a glass full with milk. (glass probably)

9. A glass filled with water or the same glass full with milk. (same)

10. An ice cream cup (dixie cup) or a pint package of ice cream. (cup)

11. An empty coffee can or a soup can full with sand. (soup can, probably)

12. A baseball or a tennis ball. (tennis ball)

Were there some exercises for which you were not sure of the answers? Why was it hard to decide? It would seem that it is usually easier to compare space regions which have the same shape.

COMPARING SPACE REGIONS (BY IMMERSION)

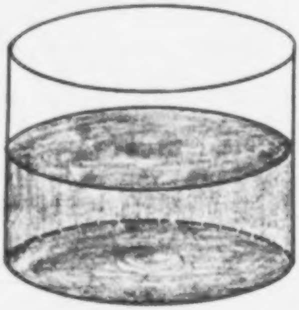
Exploration

In some of the exercises we have done it was hard to decide which one of the two given models represented the larger space region. This happened when the two models were of about the same size and shape and our eyes weren't "sharp" enough. Example: An apple and an orange of about the same size; a baseball and a tennis ball; two marbles; etc.

We also had trouble making up our minds when the two models were of such different shapes that it was hard to imagine putting one of them "inside of" the other. Example: A pear and a banana; a candy bar and a marble; etc. We shall now look at another way of using the models of two space regions to decide which model represents the larger region.

These exercises may be answered after having watched your teacher perform this experiment.

1. Take a container partly filled with water. It should be a glass or plastic one so that you can see how high the water is. Make a black mark on the container to show how high the water is.



The edges of the container and the water represent a space region. In the picture this region has been shaded. What part of the model represents the simple closed surface? The interior? Could you color the model of the boundary surface blue without coloring any part of the model of the interior? Why not?

Could you color the part of the model representing the interior of the space region red without coloring any part of the model of the boundary? Why not?

2. Take a small rock and a rubber ball. If you drop the rock into the container, will it sink or float? If you drop the ball into the water, will it sink or float?

3. If you were to drop the rock into the water, would you expect the height of the water level to change? If you answered yes, how would you expect the height to change? Why?

Now, gently lower the rock into the water and see what happens.

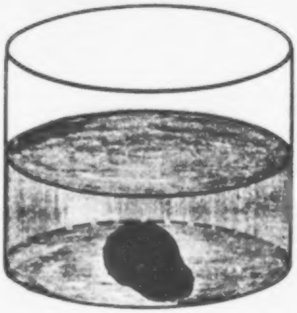


Figure 2

Make a red mark on the container to show how high the water is. Look at the red mark and the black mark. Is the water level as high as it was before? higher? not so high? Was this what you thought would happen? Remove the rock from the water.

4. The rock is a model of a space region. What part of this model represents the boundary of this region? What part of the model represents the interior of this region? Could you color the part of the model which represents the boundary blue?

5. How does the space region represented by the rock compare with the space region represented by the shaded part of figure 1? How do you know?

6. Choose a larger rock than the one you used before. If you were to put this rock into the water, how would you expect the height of the water level to change?

If you have lost any water because of splashing, fill your container up to the black mark. Now gently lower the larger rock into the water. Make a blue mark on the container to show how high the water level is.

Look at the blue mark and the black mark. Is the water as high as it was before? higher? not as high? Can you explain why?

Look at the blue mark and the red mark. When the second rock is in the water, is the water level as high as it was when the first rock was in? Is it higher? Is it lower? Can you explain why?

7. Suppose you were to put both rocks into the water and show the new height of the water level by a green mark. Without actually putting the rocks into the water, imagine what would happen and complete each of the following sentences by filling in the word "above" or "below." For example,

The black mark is below the red mark.

a. The green mark is (above) the black mark.

b. The red mark is (below) the green mark.

c. The blue mark is (below) the green mark.

Now, lower both rocks into the water, make the green mark and check your answers. Remove both rocks from the water.

8. Suppose that you have a third rock and that when you put it into the container, the height of the water level is up to the red mark.

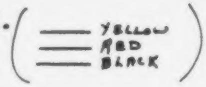
Complete the following sentences by filling in the phrases "larger than," "smaller than," or "just about as large as."

For example: The first rock is a model of a space region which is smaller than the space region represented by the second rock.

a. The third rock is a model of a space region which is (just about as large as) the space region represented by the first rock.

b. The second rock is a model of a space region which is (larger than) the space region represented by the third rock.

9. If we were to put the third rock and the first rock into the container and indicate the height of the water level by a yellow mark, draw a picture to show where the black, yellow, and red marks would be located.



10. Describe this experiment in your own words. Why do you think this is what would happen?

Try the experiment and check your drawing. Were you right?

Exercise Set 5

In each of the following sentences, cross out the extra words so that the sentence that you have is true. For example,

> If rock A is larger than rock B, the space region represented by A is (larger, ~~smaller~~) than the space region represented by B.

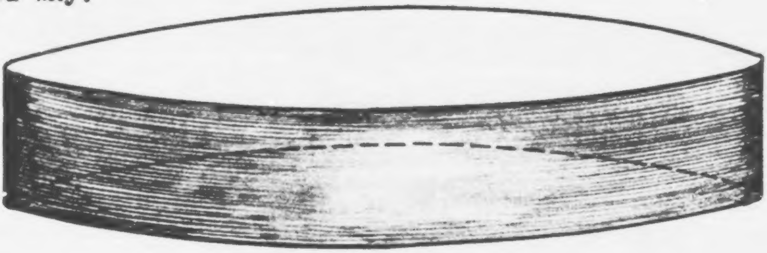
1. If a green rock is the model of a larger space region than that represented by a grey rock, putting the green rock into the container makes the height of the water level (~~lower, the same~~, higher) than putting in the grey rock.

2. When we put a red rock into the container, the water level is higher than when we put in a black rock. This helps us decide that the space region represented by the black rock is (smaller than, ~~the same size as, larger than~~) the space region represented by the red rock. Usually we just say that the black rock is (smaller, ~~the same size as, larger~~) than the red rock.

3. The higher the water level when we put a rock into the container, the (larger, ~~smaller~~) the space region the rock represents.

COMMENTS ON OUR EXPERIMENTS

You may have wondered why we wrote "just about as large as" rather than "just as large as" in the exercises you have just completed. Perhaps thinking about the following problem will show you why.



You have probably seen children's backyard pools. Some of these plastic pools are shaped like the one pictured above and are large enough for several children to play in. Suppose that such a pool is half full of water. You could make a mark on the pool to show how high the water is. If you put your first rock into the pool, will the water level rise? Why? You could make a mark to indicate the new height. How should the new mark be related to the old one? Do you think you could tell the differen
Why? Would it help to put the second rock into the pool? (Proba
you would not be able to see the difference in the water level. The rise is so small that you could not tell by looking.)

In thinking about the pool, we saw that at times, although there was a change in water level, our eye was not sharp enough to detect it. As a result, if two different rocks raise the water level to the red mark, the best we can say is that they are "about the same size." There could be a slight difference which we could not see.

In all of these experiments, we had to be careful about the kinds of objects we compared. Can you name some kinds of objects it would not be wise to compare by this water method?

MEASURE OF A SPACE REGION

To find the measure of a line segment, we use a unit line segment and see how many such units it takes to cover the given segment.

When we measure a plane region, we use a unit plane region and see how many such units it takes to cover the given region.

Can you guess how we might measure a space region?

What would be a suitable unit?

Should we use a line segment, a unit plane region, a unit space region, or some other new kind of unit?

Exercise Set 6

Tell whether you would use a unit line segment, a unit plane region, or a unit space region to get each of the following measures:

1. The size of the schoolroom floor. (unit plane region)
2. The length of a curtain rod. (unit line segment)
3. The amount of ice that can fit in a picnic ice chest. (unit space region)
4. The size of the gas tank in the school bus. (unit space region)
5. The size of a mirror. (unit plane region)
6. The size of a desk drawer. (unit space region)
7. The size of a packing carton. (unit space region)
8. The height of a door. (unit line segment)
9. The size of a tomato juice can. (unit space region)
10. The size of a chalk box. (unit space region)

We see that the measure of a space region is the number of unit space regions it contains.

How could you compare the sizes of the regions bounded by a chalk box and a milk container by using cube blocks?

In order to estimate the measure of the interior of a potato chip can, we can use small cans such as those used in packing tomato paste or tomato sauce. If each child contributes a can, there will be enough to get some measure of the potato chip can.

Why is it wise to select unit cans which have the same size and shape? How could we use these small cans to get a measure of the interior of a potato chip can? of the interior of a shoe box?

Could we use cube blocks to get a measure of the region bounded by the potato chip can? of the space region determined by the shoe box?

In each case decide which gives a better measure--the tomato juice can or the cube block. Why?

VOLUME AND ITS STANDARD UNIT

Exploration

These experiments should be demonstrated to the group by the teacher.

1. Select any two small objects of different shapes (Example: a paper cup, a fruit dish, a cereal bowl, a vase, ...) which can be used to determine models of space regions. Compare the sizes of the two regions by filling the interiors of these objects with marbles. Were all the marbles the same size? If not, explain why this is a disadvantage in comparing the measures.

2. Fill a glass with marbles. See whether the marbles actually fill the interior of the glass. Is the space region determined by a sphere a satisfactory unit for measuring the space region represented by the filled glass? Explain?

3. Could you use marbles to compare the sizes of the space regions represented by a baseball and a grapefruit? Explain your answer.

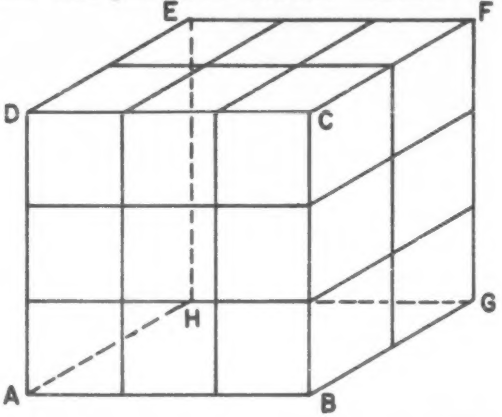
None of these ways seems to be a good way of measuring a space region. Now we are going to see if we can find a better way to get the measure of a space region using a standard unit.

Recall that when we measured line segments and plane regions we might have used any one of a variety of units. However, for purposes of effective communication, a standard unit was selected in each case. We use the inch as a standard unit of linear measure and the square inch as a standard unit of measure for a plane region.

To measure a space region, we will use as a standard unit that space region determined by a cube whose edge is 1 inch long. We call this new unit the cubic inch.

The volume of a space region in terms of this standard unit consists of (1) the measure of this region in terms of this unit and (2) the unit used.

4. Consider the box pictured below. Since 18 cubes, each of edge 1 inch long, fill the box ABCDEFGH, we say the volume of this box is 18 cubic inches.

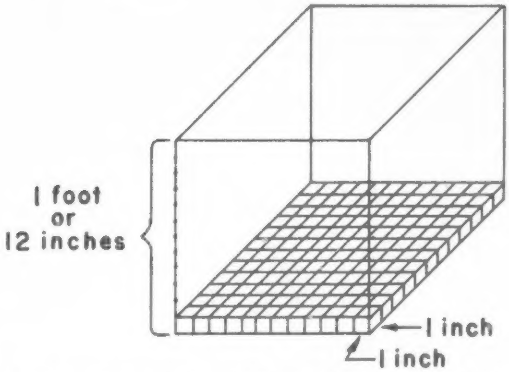


5. Use cubic inch models to estimate the volume of each of the following:

   a. A chalk box
   b. A shoe box.
   c. A rectangular baking tin.

6. It is not always reasonable to use this standard unit, the cubic inch, to find the volume. If we wished to find the volume of a refrigerator-freight car, it would be impractical to use a cubic inch as the unit of measure. For such purposes, we use a cubic foot or a cubic yard. A cubic foot is a space region bounded by a cube, each of whose edges is 1 foot long. A cubic yard is a space region determined by a cube, each of whose edges is 1 yard long.

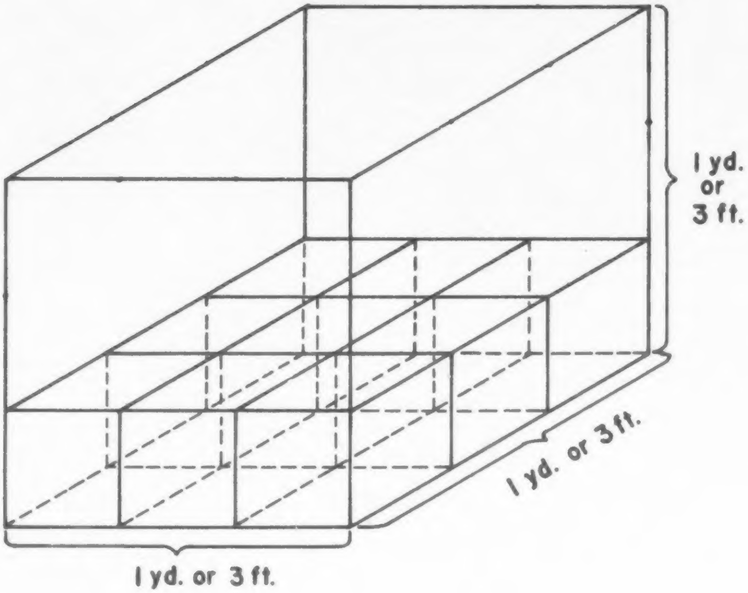
7. The sketch here should help you see the number of times that a cubic inch is contained in a cubic foot. One layer of cubic inches is sketched.
   a. How many cubic feet are there in this layer?
   b. How many layers will it take to "fill" the big cube?
   c. There are (1728) cubic inches in one cubic foot.



Because there are 12 inches in the height of the cubic foot, we could fill the interior of the cubic foot with 12 layers of the cubic inches, each layer of which is made up of 144 cubic inches. Thus, 144 x 12 = 1728 and we would need 1728 cubic inches to build a model of a cubic foot.

8. Now use the model shown below to help see the number of cubic feet in a cubic yard. Each edge of the big cube is 1 yard, or 3 feet, in length. One layer of cubic feet is sketched. How many layers will it take to fill the big cube? (3) There are 27 cubic feet in one cubic yard.

   a. How many cubic feet are there in this layer? (9)
   b. How many layers will it take to "fill" the big cube? (3)
   c. There are (27) cubic feet in one cubic yard.



9. Can you use your answers to exercises 7 and 8 to determine what the measure of a cubic yard would be in cubic inches?

There are (1728 x 27 or 46,656) cubic inches in a cubic yard.

Here is one place where computing with measures is a lot easier than counting up the number of cubic inch blocks needed to build up a model of a cubic yard. Often we can save ourselves a lot of work by using special measures to help compute a volume.

10. In measuring length we sometimes used a standard unit different from the inch, the centimeter. Using the centimeter we can obtain a unit of volume called the cubic centimeter.

   a. What sort of a space region is the cubic centimeter? (A cubic centimeter is a space region bounded by a cube, each of whose edges is 1 centimeter long.)

   b. Do you remember how many centimeters were needed to cover a length of one meter?

   There are (100) centimeters in one meter.

   c. Describe the space region we would call a cubic meter. (A cubic meter is a space region determined by a cube, each of whose edges is 1 meter long.)

   d. How many cubic centimeter blocks would we need to build a cubic meter? (1,000) How did you get your answer? (The first layer of cubic centimeters would contain 10 x 10 or 100 cubic centimeters. There would be 10 layers. 10 x (10 x 10) = 1000)

COMPUTATION OF THE MEASURE OF SPACE REGIONS BOUNDED BY PRISMS

Objective: To give the pupils understanding of the following and skills in computing measures of volume

1. Rectangular prisms of different shapes may have the same volumes.

2. To calculate the measure of a space region bounded by a rectangular prism, multiply the measures of the edges which describe the length, width, and height of the rectangular prism. The measures of the edges must be made with the same unit.

3. If the length, width, and height of a rectangular prism are in inches, the volume of the space region is in cubic inches; if the length, width, and height are in feet, the volume is in cubic feet; if the length, width, and height are in centimeters, the volume is in cubic centimeters.

4. If a, b, c are the number of units in the length, width, and height of a rectangular prism, and V is the number of cubic units in the volume of its space region, then we may write

   $$V = a \times b \times c$$

5. We may also use the statement

   $$V = (\text{measure of the base}) \times c$$

   Where V is the number of cubic units in the volume of a solid region bounded by a rectangular prism and c is the number of units in the height of the prism.

6. The upper and lower bases of any rectangular prism are bounded by congruent rectangles. The upper and lower bases of any triangular prism are bounded by congruent triangles. The upper and lower bases of any prism are bounded by congruent polygons.

7. For any of the prisms considered in this unit, the statement

   V = (measure of the base) × (number of units in the height)

   is true.

8. The measure of a plane region bounded by a triangle is calculated by multiplying the number of units in the base of the triangle by the number of units in the altitude to that base, and dividing the product by 2.

9. Two simple closed surfaces are congruent if either can be made to fit into the other. Two congruent simple closed surfaces bound space regions of equal volume.

Materials:

Teacher:	Cube blocks, cardboard models of rectangular prisms (shoe box, milk carton, pill box, etc.), scissors, paper for making model of triangular prism, scotch tape.
Pupils:	Same as teachers.

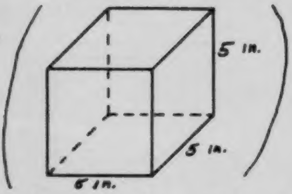
Vocabulary:

Obtuse triangle, acute triangle, polygon

Exercise Set 7

1. Suppose now that we have a cubical chalk box, each edge of which is 5 inches long.

   a. Draw a picture of this box.

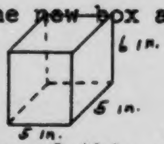


   b. What is the volume of this chalk box in cubic inches? (125)
   How do you know? (One layer would contain 25 cubic inches. There are 5 layers. 5 × 25 = 125)

   c. Can you describe two ways of solving this problem?
   (You could imagine horizontal layers or vertical layers.)

2. Suppose that we have a second chalk box which is one inch higher than the first.

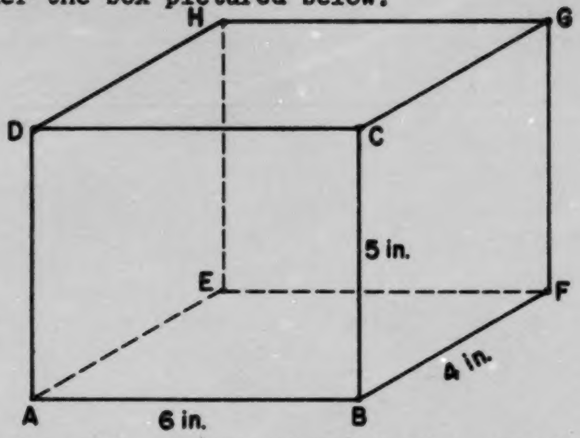
   a. Draw a picture of this box. In your own words, tell which edges of the new box are longer than those of the first box. (The vertical edges are each 1 inch longer than those of the first box.)



   b. What is the volume of this new box in cubic inches?
   ((5×5) × 6 = 150)

   c. Describe two ways of finding this volume. (Using 6 horizontal layers of 25 cubic inches each or 5 vertical layers of 30 cubic inches each.)
   Which one did you use?
   (Answers will vary.)

3. Consider the box pictured below.



Its surface is a simple closed surface which together with its interior describes a space region. Edge AB is 6 inches long and is usually called the length of the box. Edge BF is 4 inches long and is called the width of the box. Line segment BC, 5 inches long, is called the height of the box.

a. The chalk box is a model of a figure with (6) faces and (12) edges. Each face is a (plane) region bounded by a (rectangle). The surface represented by the chalk box is called a (rectangular prism).

b. Three segments congruent to $\overline{AB}$ are represented by the edges ($\overline{EF}$), ($\overline{DC}$), and ($\overline{HG}$). Each of these is (6) inches long.

c. The edges ($\overline{AE}$), ($\overline{DH}$), and ($\overline{CG}$) represent segments each of which is congruent to $\overline{BF}$. The measure in inches of each of these edges is (4).

d. The segment BC is congruent to the three segments represented by the edges ($\overline{GF}$), ($\overline{EH}$), and ($\overline{AD}$). All 4 of these segments have a common length of (5 in).

e. Face ABCD has area (30) square inches.

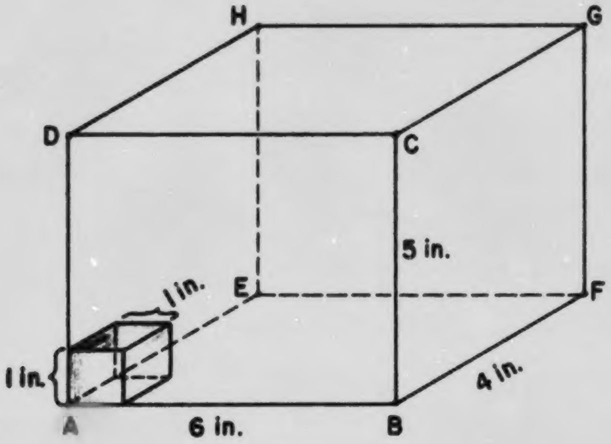
Another face congruent to ABCD is (EFGH).

f. Face BFGC has area (20) square inches. (AEHD) is another face congruent to BFGC.

g. Two other congruent faces are (ABFE) and (DCGH). Each of these has area (24) square inches.

h. Is the box a cube? (No). How can you tell? (All edges are not the same length.)

4. Think of putting cubic inch blocks on the floor (face ABFE) of the chalk box of exercise 3. Each block is placed so that it has a face touching the floor.

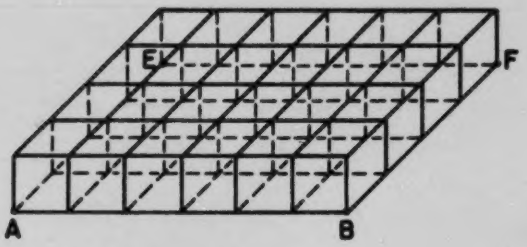


a. The face of the block which touches the floor is a plane region bounded by a (square). Each of its edges has a length of 1 inch.

b. The area of this face is (one) square inch.

c. The area of the floor is (24) square inches.

d. If you fit the blocks as tightly as you can, you would need (24) blocks to cover the whole floor.



e. We would need (5) layers of blocks like the one on ABFE to fill the whole box.

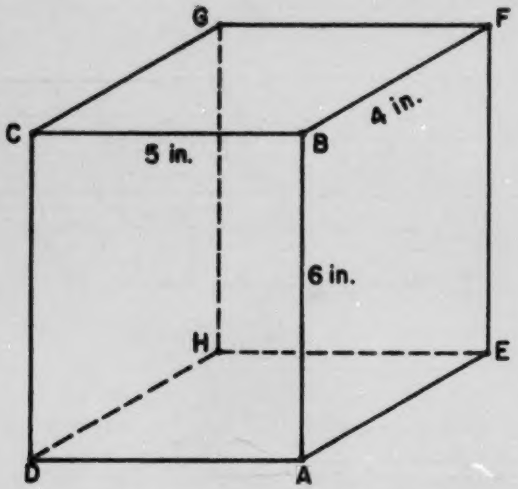
f. Each such layer is built out of (24) blocks.

g. We would need (120) blocks to fill the box.

h. The measure in cubic inches of the space region bounded by the chalk box is 120.

Its volume is (120 cu. in.).

5. Suppose we now stand the chalk box on another face, say face DAEH.



a. The measure of DA, the length of the box, in inches, is (5).

b. The measure of the width, AE, in inches is (4).

c. The height, AB, in inches is (6).

d. You see that the edges we call the length, width, and height of the box are determined by the face the box is resting on.

Do you think the volume of the region described by the box is the same as it was when the box was resting on face ABFE? (Yes) Why? (Position does not change the volume.)

6. a. When the box of exercise 5 is in this position, its floor is face (DAEH).

b. Once again let us imagine that we are covering the floor of the box with blocks which are models of the cubic inch. It will take (20) blocks to cover the floor.

c. Is this the same number of blocks we used before? (No).

d. Do you now think the volume of the box is the same as it was when it rested on face ABFE? You can change your mind if you would like to. (Yes). Why? (Position does not change the volume.)

e. We will need (6) layers of blocks just like the one used to cover the floor to fill the entire box.

f. There are (20) blocks in each layer. Therefore, we need (120) blocks to fill the whole box.

g. This tells us that the measure in cubic inches of the region bounded by the box is 120.

h. The volume of this region is (120 cu. in.).

i. How does this volume compare with the volume we got when the box was resting on face ABFE? (It is the same.)

7. a. Do you think that if we let the box rest on face ABCD the volume would be changed? (No). Why? (Position does not change the volume.)

   b. Tell how you could use layers of cubic inch blocks to show that your answer is correct. (30 blocks per layer - 4 layers)

8. a. Do you see a quick way to find the volume of the chalk box using the measures in inches of the length, width, and height?

   b. Does your way depend upon which face is the floor of the box? (No)

   c. Can you explain why your way works?

   d. Say in words what measures you would like to know to find the volume of a space region bounded by a rectangular prism. (the measure of the length, width, and height)

9. a. Is a cube a rectangular prism? (yes)

   b. Is every rectangular prism a cube? (No) Why? (all edges must be congruent segments in order for it to be a cube. This will not be true for every rectangular prism.)

   c. How could you find the volume of a cube? (Since each edge must have the same measure, then $V = e \times e \times c$.)

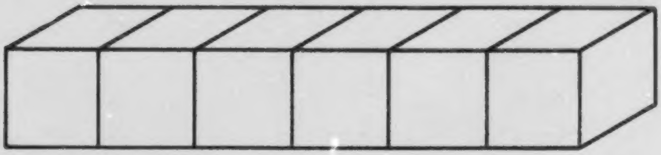
   d. Tell in words which measures you would like to know to find the volume of a cubical space region. (you need to know the measure of one edge.)

   e. Is this an easier or a harder problem than finding the volume determined by a box which is not a cube? Why?

Exploration

The volume of the space region bounded by a prism is often just called the volume of the prism. In the same way we often speak of the volume of a cube when we really mean the volume of the space region bounded by the cube. We can use the shorter expression whenever we are fairly sure that we will not be misunderstood.

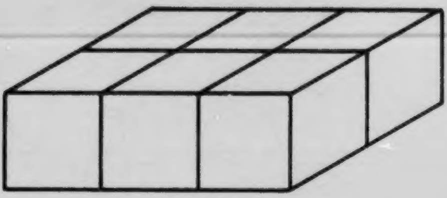
1. If we place 6 cubes as in the diagram below, we have built a model of a rectangular prism.



   a. What is its volume? Observe that this model is 6 units long, 1 unit wide, and 1 unit high. (6 cu. unit)

   b. Are the units we refer to linear units, area units, or volume units? (When referring to the length, width, and height we mean linear unit. If we refer to the area of the base, then we mean area unit. If we refer to volume, then we must use cubic unit.)

2. Another model of a rectangular prism which could be made from the same six cubes would look like this:



a. What is the volume of the above model? (6 cubic unit)

b. What are the length, width, and height? (length is 3 unit
width is 2 unit
height is 1 unit)

c. Can you build still another model of a rectangular prism using these six cubes? (yes, there are several possibilities)

d. What are the length, width, and height of your new rectangular prism? (length is 2 unit
width is 1 unit
height is 3 unit
Answers will vary)

e. The length, width, and height are expressed in what kind of units? (linear unit)

f. The volume, however, is expressed in (cubic) units.

3. Choose 24 unit cubes and build at least 6 different models of space regions bounded by rectangular prism. Keep a record of the following measures. Remember that the length, width, and height are measured in one kind of unit and the volume in another. (There are several possibilities. The following are some examples.)

	Measure of Length	Measure of Width	Measure of Height	Measure of Volume
a.	2	3	4	24
b.	1	4	6	24
c.	8	3	1	24
d.	3	4	2	24
e.	24	1	1	24
f.	2	12	1	24

g. Do you see any relationship between the measures of the length, width, and height of the prism and the measure of its volume? (The product of the measures of the length, width, and height is always 24.)

h. Was this what you expected? (Yes)

i. Can you explain what you observed? (The measures of the length, width, and height may change but this does not always mean that the volume will change.)

Exercise Set 8

1. List 5 sets of possible lengths, widths, and heights of rectangular prisms whose volume is 36 cubic inches. Do not actually build models of these prisms.
(Answers will vary. Some examples are 2,2,9; 3,3,4; 6,6,1; 12,3,1; 36,1,1.)

2. List 5 sets of possible lengths, widths, and heights of rectangular prisms whose volume is 144 cubic feet.
(Answers will vary. Some examples are 4,6,6; 12,12,1; 3,6,8; 9,4,4; 6,2,12.)

3. Think of a cube whose edge is 1 centimeter long. What do you think would be a suitable name for this unit of measure of a space region? (cubic centimeter)

   List 5 sets of possible lengths, widths, and heights of rectangular prisms whose volume is 100 cubic centimeters. (Answers will vary. Some examples are 10,10,1; 2,10,5; 25,2,2; 5,5,4; 1,2,50.)

4. If the length, width, and height are in inches, the volume is in (cubic inches). If the length, width, and height are given in feet, the volume is in (cubic feet). If the length, width, and height are given in centimeters, the volume is in (cubic centimeters).

5. Complete the following table. All measures refer to space regions bounded by rectangular prisms.

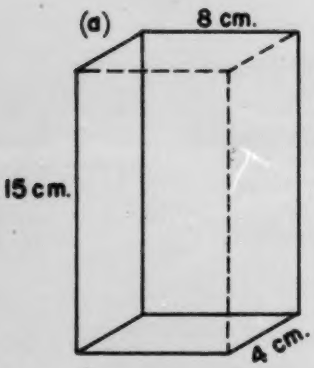
	Measure of Length of Prism	Measure of Width of Prism	Measure of Height of Prism	Measure of Volume of Space Region
a.	5	8	3	(120)
b.	2	6	8	(96)
c.	12	4	1	(48)
d.	(4)	6	7	168

6. To calculate the measure of a solid region bounded by a rectangular prism, multiply the measures of the length, width, and height of the rectangular prism. If a, b, c, are the number of units in the length, width, and height of a rectangular prism, the volume of the prism is $(a \times b \times c)$.

7. Show how you could use an exponent to write the product for calculating the volume of a cube of edge 5 inches, (5^3) of edge 2 feet, (2^3) of edge a centimeters. (a^3)

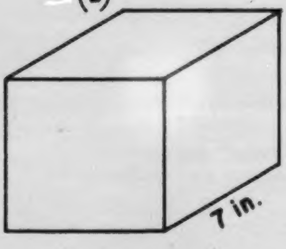
8. Calculate the volume in each case, if enough information is given.

(a)



(480 cu. cm.)

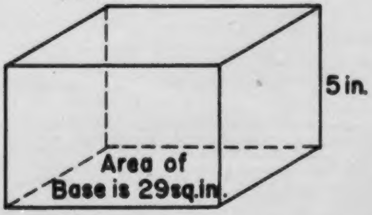
(b)



This figure is a Cube

(343 cu. in.)

(c)



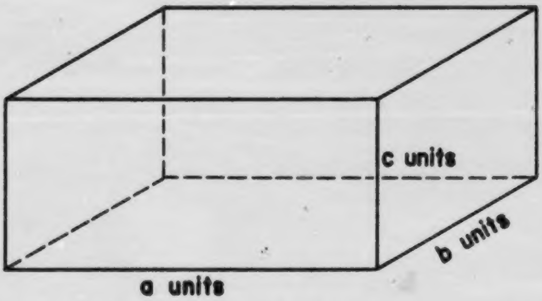
(145 cu. in.)

COMPUTING THE VOLUME OF A SPACE REGION BOUNDED BY A RIGHT PRISM (BOX)

Exploration

Were you able to calculate the measure of the space region in exercise 8 (c) of the last set of exercises? Here is one way you might have thought about this probler.

If the measure of the length, width, and height of a rectangular prism are a, b, and c, then the measure of the volume is $a \times b \times c$. The measure of the length, width, and height is, of course, the number of linear units needed to cover each of these line segments. The measure of the volume, however, tells us how many cubic units there are in the space region bounded by the prism. All four of these measures are numbers. If we let V stand for the measure of the volume, we can write what we have discovered as $V = (a \times b) \times c$.



The product $(a \times b)$ gives the measure of the rectangular region which is the base of the prism. Another way of saying this is that $a \times b$ tells us how many units we need to cover the base of the prism. Therefore, in problem 8 (c) we may write:

$V =$ (measure of the base) $\times c$
$V = 29 \times 5$
$V = 145$

The volume is 145 cubic inches.

Exercise Set 9

1. Calculate the volume of a space region bounded by a rectangular prism, using the following information:

	Area of Base	Height	Volume
a.	42 sq. cm.	15 cm.	(630 cu. cm.)
b.	38 sq. in.	10 in.	(380 cu. in.)
c.	100 sq. in.	4 ft.	(4800 cu. in.)

NOTE: 4 ft. = 48 in.

2. If the volume of a solid region bounded by a rectangular prism is 144 cubic units and the height 10 linear units, what is the area of the base? (14.4 sq. unit)

3. How many packages of paper napkins can be packed in a carton whose base is a 2 ft. square, if the carton is 3 ft. high? Each package of paper napkins is 6 in. by 6 in. by 4 in.

($V_{carton} = 6$ cu. ft.

$V_{napkin} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{12}$ cu. ft.

$6 \div \frac{1}{12} = 72$

72 packages can be put in the carton.

OR

$V_{carton} = 288 \times 36 = 10,368$ cu. in.

$V_{napkin} = 6 \times 6 \times 4 = 144$ cu. in.

$10,368 \div 144 = 72$

72 packages can be put in the carton.)

4. Space is needed in a classroom to hold a set of textbooks. Each textbook is one inch thick and has a cover 6 in. by 8 in. Can a space 1680 cu. in. hold 35 texts?

(Volume of one textbook equals 48 cu in.

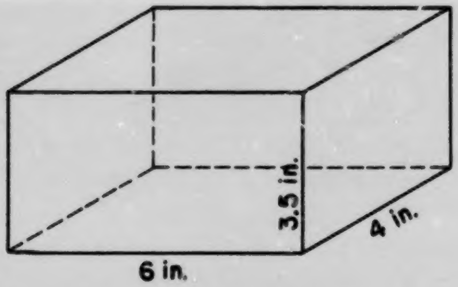
Volume of 35 textbooks equals 1680 cu. in.

It would be most convenient if the length was 35 inches, the width 6 inches, and the height 8 inches. Of course some other dimensions would work.)

Exploration

Let us look at a box where the length of one edge is not a whole number of units.

1. Look at the box pictured below.



Its length is (6 in.)

Its width is (4 in.)

Its height is (3.5 in.)

a. If you use blocks which are models of the cubic inch, how many such blocks would you need to cover the floor? (24)

b. The area of the floor of the box is (24 sq. in.).

c. If you used two such layers of blocks, would you exactly fill the box? (No)

d. Would two layers contain too many blocks, just enough blocks, or too few blocks to fill the box? (too few)

e. This tells us that the volume of the box is (more than) 48 cubic inches.

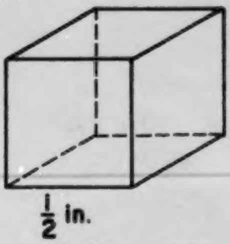
f. Would 3 such layers exactly fill the box? (No)

g. Would 4 such layers exactly fill the box? (No)

h. We see that the volume of the box is (more than) 72 cubic inches and (less than) 96 cubic inches. This is the best we can do using cubic inch blocks.

i. What sort of blocks could we use to get a better estimate of the volume of this box? (smaller ones)

2. Suppose we have a cubical block of edge $\frac{1}{2}$ inch.



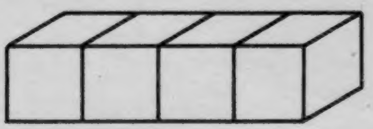
a. The volume of the space region for which this block is a model is $(\frac{1}{8}$ cu. in.) .

b. It would take (8) of these blocks to build a model of the cubic inch.

3. a. Let us use these smaller blocks to cover the floor of the box of exercise 1. We will pack these blocks in tightly just as we did before and in such a way that each block has a face on the floor. The face of the block is a model of a (plane region) bounded by a (square) of area $(\frac{1}{4}$ sq. in.) .

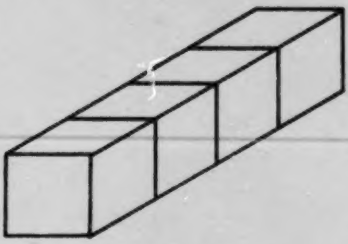
b. We would have to pack in (4) such blocks to cover one square inch of the floor.

4. a. If we line the blocks up as shown below,



it would take (12) of these $\frac{1}{2}$ inch blocks to build a model as long as the box.

b. If we build columns of blocks like this,



it would take (8) such blocks to build a prism as wide as the box.

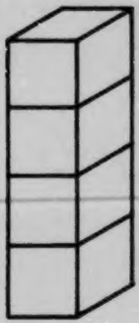
c. Using these facts we see that it would take (96) of our new blocks to cover the entire floor of the box.

d. Every 4 such blocks would cover (one) sq. in. of the floor.

e. This shows that our 96 blocks will cover (24) sq. in.

f. The area of the floor of the box is (24 sq. in.).

5. a. If we pile up blocks like this,



it will take (7) such blocks to build a prism as high as the box.

b. This helps us conclude that we will need (7) layers of blocks just like the one on the floor to fill the whole box.

c. Each such layer contains (96) blocks and therefore we will need (7 × 96) or (672) blocks to fill the box.

d. The measure of our box in terms of this new unit is (672)

e. If we now remember that it took 8 of these blocks to build a model of (one) cu. in., we see that the volume of the box is (672 ÷ 8) cu. in. or (84) cu. in.

6. We also see that this volume of the box of exercise 5 is less than 96 cubic inches and bigger than 72 cubic inches. This was pointed up by the discovery that 3 layers of 24 cubic inch blocks each were not quite enough to fill our box, while if we built 4 such layers we had too many blocks.

a. Using these smaller blocks we can get (a better / a more accurate / an exact) estimate of the volume.

b. In general, using a smaller unit will lead to a more (accurate) estimate.

c. For the box in question, the measure (in inches) of the length, a, is (6).

d. The measure (in inches) of the width, b, is (4).

e. The measure (in inches) of the height, c, is (3.5).

f. a × b × c is (84). Is this the measure, V, (in cubic inches) of the volume? (Yes)

g. Will our formula V = a × b × c work for this box? (Yes)

Exercise Set 10

1. Consider a box of length $6\frac{3}{4}$ inches, width 4 inches, and height 3 inches.

   a. Draw a picture of this box.

   b. What are a, b, and c for the prism represented by this box?

   a = $(6\frac{3}{4})$ b = (4) c = (3)

   c. Our formula leads us to believe that V = 81 and that the volume of the space region bounded by this prism is (81 cubic inches).

2. Could you check your answer by building up layers of cubical blocks of edge 1 inch? Explain. (we could find that the volume would be between 72 cubic inches and 84 cubic inches.)

3. Could you check your answer by building up layers of cubical blocks of edge $\frac{1}{2}$ inch? Why? (we could find that the volume would be between 78 cu. in. and 84 cu. in.)

If we studied more rectangular prisms, we would find that we could always use the formula, $V = a \times b \times c$ to help us find the measure of the volume.

Another way of thinking of this formula is

$$V = (a \times b) \times c \quad \text{or}$$

Measure of volume (in cubic inches) = Measure of base (in square units) × height (in linear units)

VOLUME OF SOME SPACE REGIONS WHOSE BASES ARE NOT RECTANGLES

In this section you will find some reference made to the volume of a cylinder in the pupil text. Finding the measure for the volume of a cylinder is dependent upon finding the measure for the area of the base. The base of the cylinder considered here is a circular plane region. We do not now have a way of finding the area of a circular region. But it is useful to help the pupils see that *if* they could find the area of the circular region that is the base of the cylinder, then they could find the volume of the cylinder.

The method of approximating the area of the circular region by using the grid is suggested as a method for finding two numbers between which the measure of the area is likely to be. The result found gave us an estimate of the measure of the volume between 50 and 65. From your own knowledge of geometry you know that the area of the circular region is πr^2 where π represents an irrational number (about 3.1416) and r is the radius of the circle. This would give the measure of the volume as

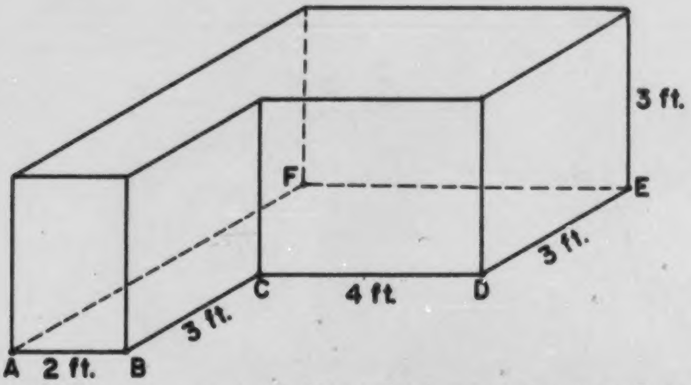
$$3.1416 \times 4 \times 5 = 62.8320,$$

but this is not to be presented as a part of this chapte[illegible] You may wish to indicate that they will come to this result later in their formal study of geometry.

VOLUME OF SOME SPACE REGIONS WHOSE BASES ARE NOT RECTANGLES

Exploration

1. a. Consider the space figure pictured below. (A model of such a figure might be a storage chest specially designed to fit into a corner of a room.) Can we find the volume of the region bounded by this figure?



b. Since the measures of the edges are expressed in terms of feet, what unit would we expect to use to express the volume? (cubic feet)

c. One model of this unit would be a cubical block. What would be the length of the edge of such a cube? (one foot)

d. If we pack these cubes as usual, how many such cubes would it take to cover the floor of such a chest? (24)

e. If you have trouble answering this question, you might imagine making a trace of the outline of the floor of the chest. Your trace would look like the figure at the top of the next page.

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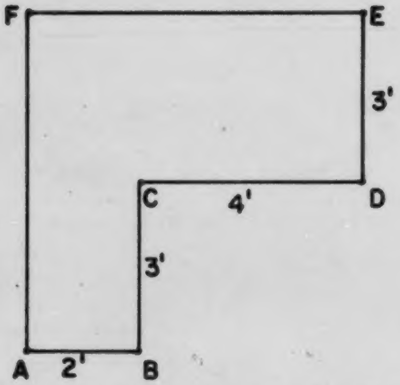


2. Look at the outline shown above of the floor of the chest.

a. How long is $\overline{AF}$? (6 ft.)

b. How long is $\overline{FE}$? (6 ft)

c. Does this help you answer your question?

d. The area of the floor of the chest is (24 sq. ft.).

e. How many such layers of blocks would you need to fill the box? (3 layers)

f. The measure of the volume of the chest is (24 × 3) (in cubic feet) or (72) (in cubic feet).

V, the measure of the volume, is the measure of the base (in square feet) multiplied by the measure of the height (in feet).

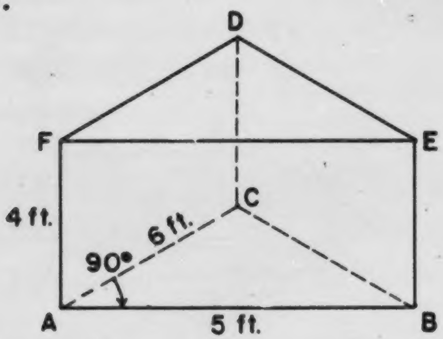
$$V = B \times h$$

This was the same formula that we used for the chalk box. The base of the chalk box was a plane region bounded by a rectangle whose sides had measures a and b where a and b represent numbers. The measure of the base (in square units), therefore, was $a \times b$. If the height of the box is c, $V = B \times h$ becomes $V = (a \times b) \times c$. The letters a, b, c represent numbers.

3. A storage container is pictured below. It is a model of a space region.

   a. The base of the space region is a (plane region) bounded by a (right triangle).

   b. If again we use blocks which are models of the cubic foot, we will need (4) layers of such blocks to fill the container.



   c. Can we exactly cover the floor using these blocks? (No, unless we are able to cut them up and repack.)

   d. What is the area of the plane region which has the floor of the chest as its model? (15 sq. ft.)

   e. Could you cover this floor of the chest using your cubic-foot blocks? (Yes, if you can saw them up and repack them.)

   f. Can you imagine such a covering?

   g. How many blocks would you need? (15)

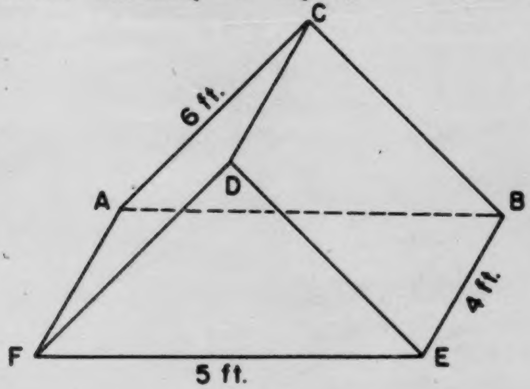
   h. To fill the entire container we would need how many blocks (60)

   i. Does our formula, $V = B \times h$, work for this space figure? (yes)

   j. What is the volume of the region described by the picture? (60 cubic feet)

4. a. Face ABEF of the container is a model of a plane region bounded by a (rectangle).

b. Suppose we let the model rest on this face. Do you think the region now bounded by this figure will have a different volume? (No) Why? (It should hold as many cubic feet.)



5. a. If the container is in this position, its floor is the model of a plane region of area (20 sq. ft.).

b. Could we use

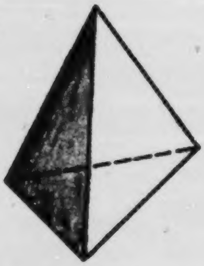
$$V = Bh = 20h$$

where h is the measure (in feet) of the height from the base up to edge DC to find the measure of the volume? (No) Why? (We can't build a model of this space region using equal layers of blocks. We would need fewer blocks per layer as the layers get farther from the bottom.)

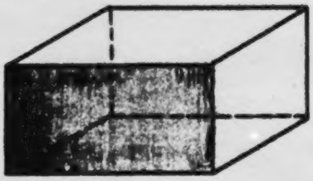
Exercise Set 11

1. Which one of the following pictures represent space regions the measure of whose volumes could be found by V = Bh? Why? (b, e, and g. The others do not have a plane region congruent to the base.)

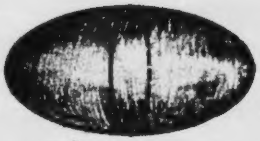
a.



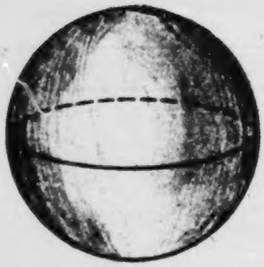
b.



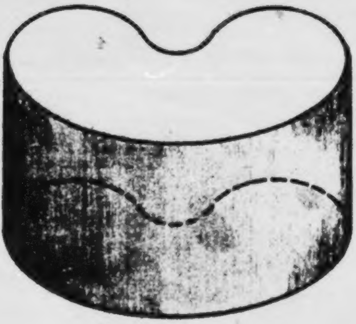
c.



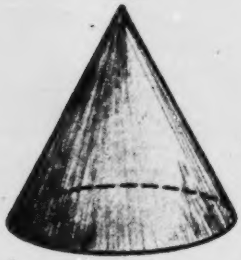
d.



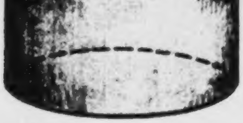
e.



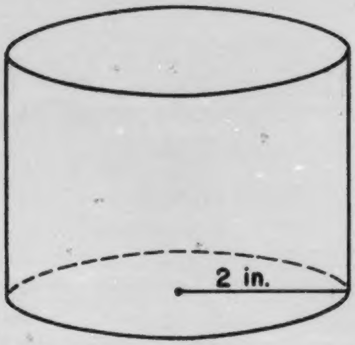
f.



g.



2. Consider the space region pictured below.



a. A model of such a space region might be a tin can. Do you think you could find the measure of the volume of this region by means of the formula, $V = Bh$? (Yes, but only if we could find a measure of the area of the base in square units. We do not know how to do this now.)

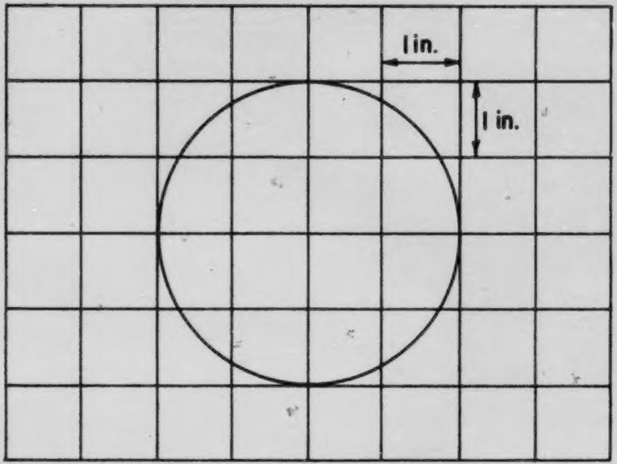
b. How did you make your decision? (We could build up the model by using equal layers of blocks where one block just covers the base.)

c. What unit would you expect to express this volume? (cu. in.) Why? (All linear measures were expressed in inches.)

d. What is the name of the surface bounding this space region? (Cylinder)

e. What is measure of h, in inches, for this region? (5) (h represents length)

f. One way of estimating B is to use a rectangular grid as we did in the unit on area. We could use our model of the region and make a trace of the boundary of its base. This boundary curve is a circle of radius 2 inches. Your trace would look like this:



A grid with 1 inch squares has been put on your trace. What is the measure (in square inches) of the plane region bounded by a circle of radius two inches? (Use your grid to estimate this measure.

g. Using this grid, our smallest estimated value of B is (10) and the value of V that goes with it is (50).

h. Our largest estimated value of B is (13) and the value of V that goes with it is (65).

i. The volume of the space region is smaller than (65 cu. in.) and greater than (50 cu. in.).

j. How could we get a better estimate of the volume?
(Use a smaller mesh to get a more precise value of B.)

We have seen that some space regions -- those whose models can be built up out of equal layers -- are such that the measure of their volume is given by

$$V = B \times h$$

or

Measure of Volume (in cubic units) =

Measure of Base (in square units) ×
Measure of height (in linear units)

There are other space regions the measure of whose volume cannot be found this way.

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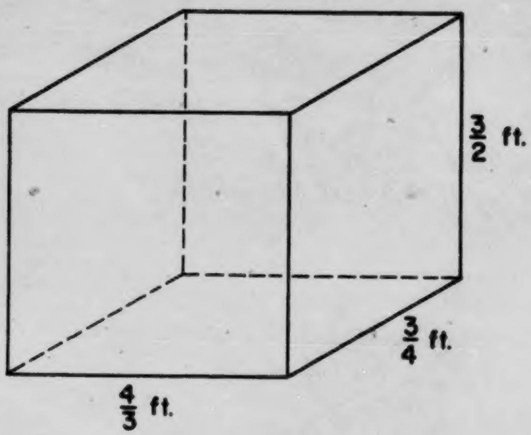
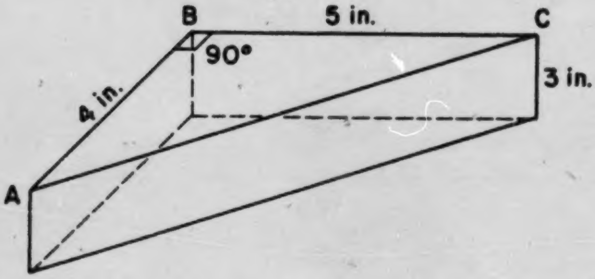
Exercise Set 12

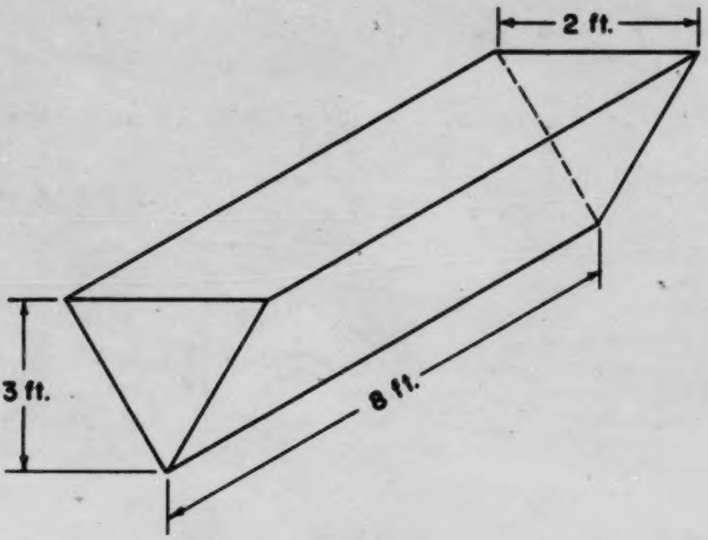
1. A hat box is a model of a space region. What is the volume of this region? $(\frac{3}{2}$ cu. ft.)


$\frac{3}{2}$ ft.
$\frac{3}{4}$ ft.
$\frac{4}{3}$ ft.

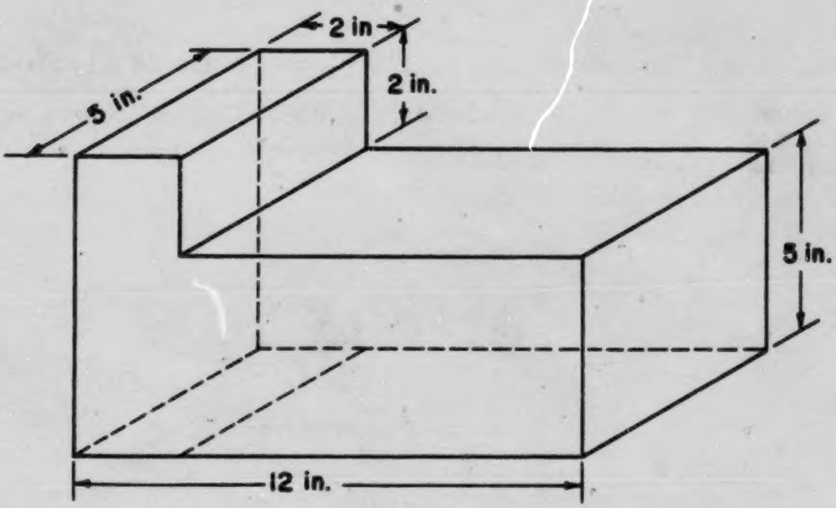
2. A plastic dish which can hold a piece of pie is pictured below. What is the volume of the space region represented by this dish? (30 cu. in.)


B
5 in.
C
90°
3 in.
4 in.
A

3. What is the volume of the water trough pictured below? (24 cu. ft.)



4. Find the volume of the space region bounded by this space figure. (320 cu. in.)

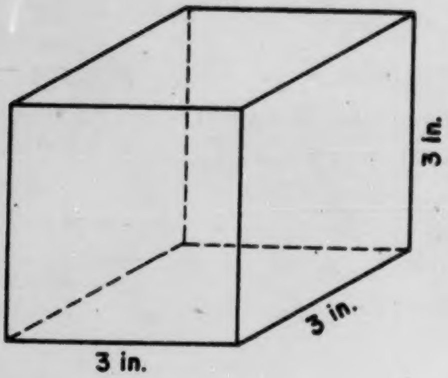

2 in
2 in.
5 in.
5 in.
12 in.

COMPARISON OF VOLUMES OF SPACE REGIONS BOUNDED BY RECTANGULAR PRISMS AND PYRAMIDS

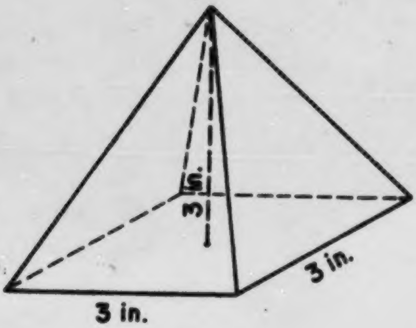
Objective: To reach the conclusion that if a pyramid has its base congruent to the base of a cube and its height equal to the height of the cube, then the measure of the volume of the space region bounded by the pyramid is one-third of the measure of the volume of the space region bounded by the cube

Teaching Procedure:

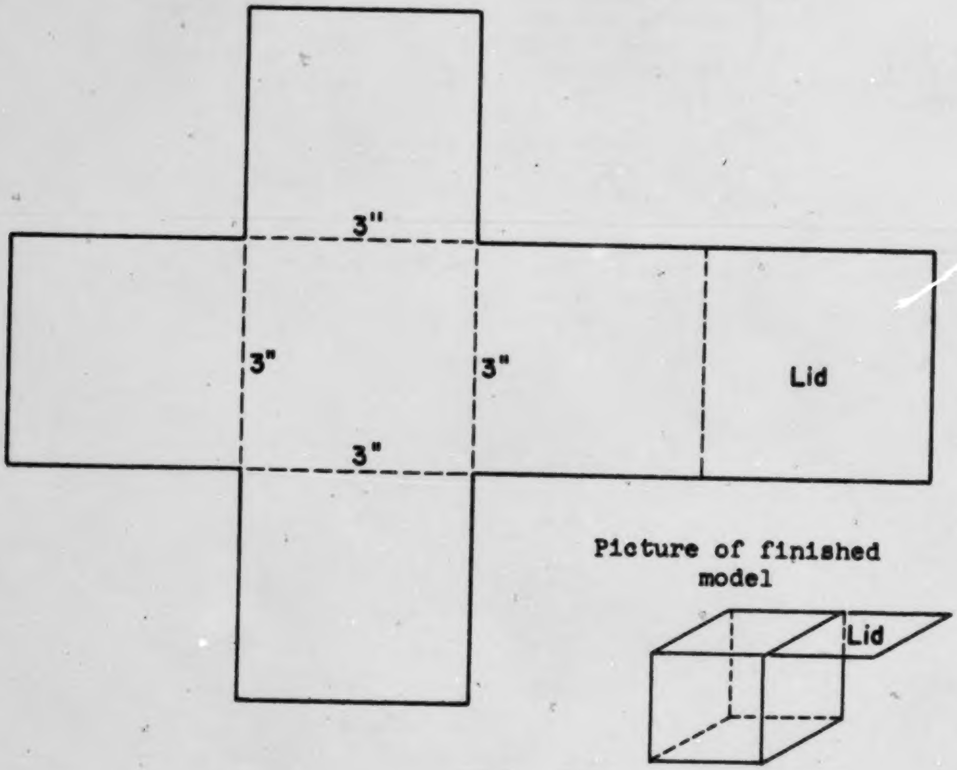
> Use the models on the following pages for cut-outs to make cubes and pyramids. For convenience, you may wish to make them larger than the models suggested. To do this, multiply each length by 2 (or 3, or whatever factor is convenient). The segments of length 3 inches, 3.67 inches would thus become 6 and 7.34, respectively, and if measurements are made as carefully as possible, useable models should result. Fasten the appropriate edges together with strong tape. The suggestions made in the pupil exploration section should be followed as explanation is made. Better results will likely result from teacher demonstration than from pupils using the models.



CUBE

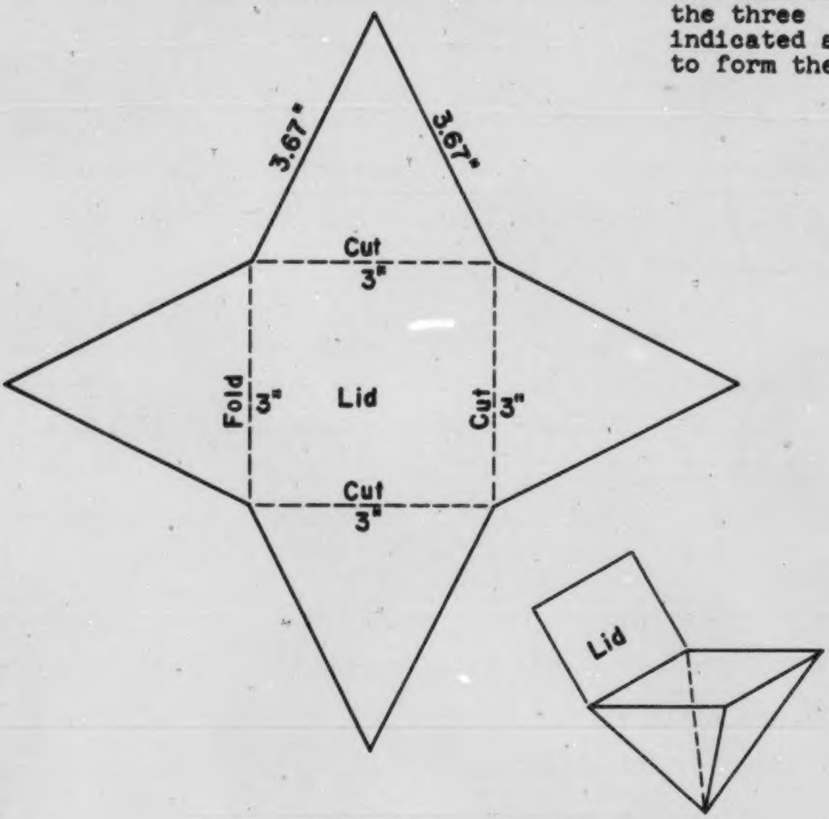


PYRAMID


3"
3"
3"
3"
Lid
Lid

Picture of finished model

NOTE: Assemble the model and then cut along the three indicated segments to form the lid.



Each side of square is 3 inches long. The height of each isosceles △ is 3.35 inches, and each of the congruent sides is 3.67 inches long. This will make the height of the pyramid 3 inches.

These models will make it possible for 3 pyramids of sand to fill the cube.

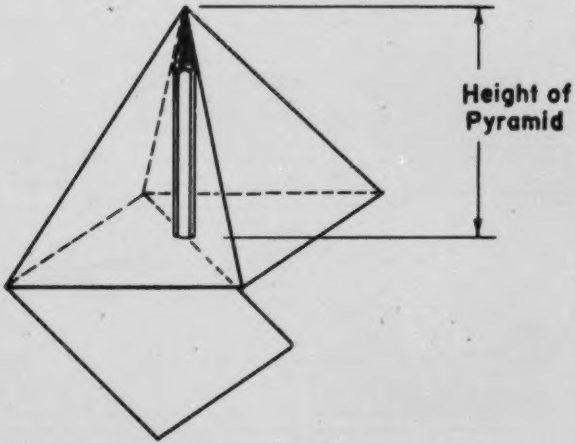
COMPARISON OF VOLUMES OF SPACE REGIONS BOUNDED BY RECTANGULAR PRISMS AND PYRAMIDS

Exploration

1. Now look at the pyramid which your teacher has. Find the linear measures (in inches) of the sides of the base of this model. Record your measures under the first two headings in the table below.

Measure (in inches) of the length of the base	Measure (in inches) of the width of the base	Measure (in inches) of the height of the pyramid	Measure (in cubic inches) of the volume

2. What is the height of the pyramid? Your teacher will now show you how to estimate it. (Let the base pieces hang free as in the picture below so that you can use a lollypop stick, a pencil, or any other thin, pointed object to help estimate the height of the pyramid.)



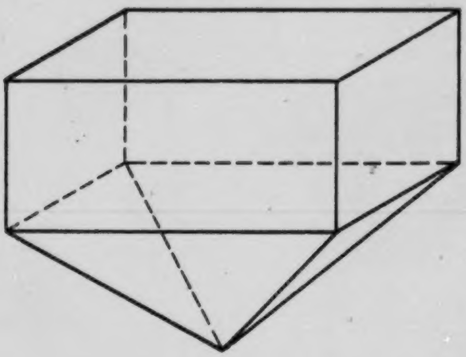
3. a. How do the measures of the cube compare with those of the pyramid?

   b. Which space region do you think is larger, that bounded by the prism or that bounded by the pyramid? (that bounded by the prism)

   c. Your teacher will fill the pyramid with rice. How many pyramids of rice does it take to fill the interior of the cube? (3)

   d. Can you estimate the volume of the pyramid? Record this estimate under the fourth heading in your table.

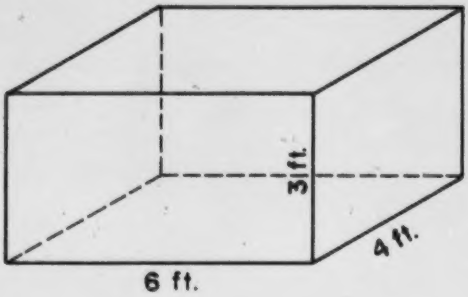
4. If a rectangular prism and a rectangular pyramid have congruent bases and heights of equal measure, how do you think the volumes bounded by these surfaces will compare?

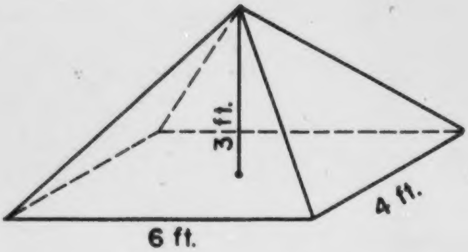


5. a. We have performed an experiment which showed that if a prism and a pyramid have heights of the same measure and if their bases are bounded by congruent rectangles, the volume of the (prism) is three times the volume of the (pyramid).

   b. Another way of saying this is that the volume of the (pyramid) is one-third the volume of the (prism).

6. Consider the prism and the related pyramid pictured below.





a. Since the length, width, and height are expressed in feet, we will expect to express the volume in (cubic feet).

b. The heights of the two figures are (3 ft.).

c. The bases of the two figures are (24 sq. ft.).

d. The volume of the pyramid is (one-third) the volume of the prism.

e. Another way of saying this is that the volume of the prism is (3 times) the volume of the pyramid.

f. The volume of the prism is (72 cu. ft).

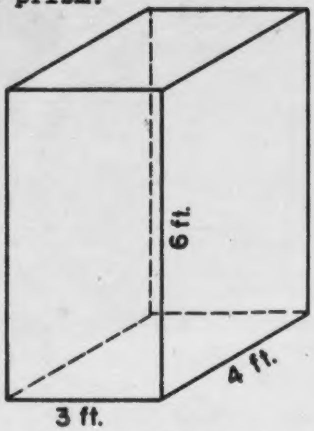
g. From this we can conclude that the volume of the pyramid is (24 cu. ft.).

7. a. We know that if a, b, and c are the measures of the length, width, and height of a prism, the measure, V, of the volume is given by the formula $V =$ ($a \times b \times c$).

b. If we have a pyramid with length of measure a, width of measure b, and height of measure c, the measure of its volume will be given by the formula $V = \frac{1}{3} \times a \times b \times c$.

c. Tell what this formula says in words.

8. Now consider this prism.



a. How does the volume of this prism compare with the one we had before? (same)

b. Does this seem reasonable? (answers will vary.)

c. How does the volume of our pyramid compare with the volume of this prism? (one-third of the volume of the prism)

d. What are a, b, and c for the new prism? ($a = 3$, $b = 4$, $c = 6$)

e. For the prism of exercise 6? ($a = 6$, $b = 4$, $c = 3$)

f. For the pyramid of exercise 6? ($a = 6$, $b = 4$, $c = 3$)

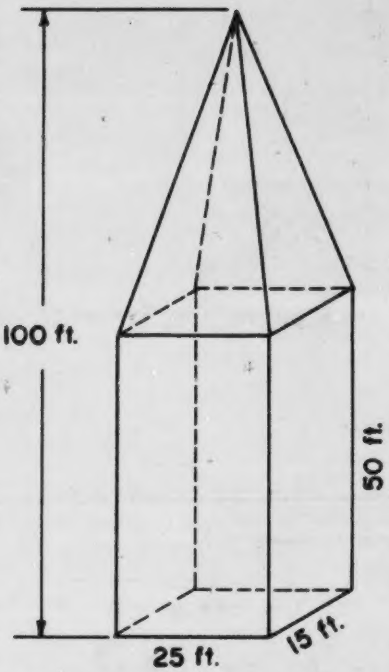
g. Does $V = abc$ give you the measure of the volume of each prism. (Yes)

h. Does $V = \frac{1}{3} a \times b \times c$ give you the measure of the volume of the pyramid? (yes)

10. What is the volume of a pyramid of length 10 inches, width 4 inches, and height 1 foot? Express your answer in terms of cubic inches. ($V = \frac{1}{3} \times 10 \times 4 \times 12 = 160$ cu. in.)

Exercise Set 13

1.



What is the measure
in cubic feet
of this monument?

$V_{prism} = 25 \times 15 \times 50 = 18,750$ cu. ft.

$V_{pyramid} = \frac{1}{3} \times 18,750 = 6,250$ cu ft.

$V_{monument} = 18,750 + 6,250$
$= 25,000$ cu ft.

2. The volume of a rectangular prism is 216 cubic inches. The height is 9 inches.

   a. What is the area of the base? (24 sq. in.)

   b. Name some pairs of numbers which could be its length, and width, in inches. (answers will vary.)

3. The volume of a rectangular pyramid is 216 cubic inches. The area of the base is the same as that of the prism of exercise 2. What is the height of this pyramid? (9 inches)

Chapter 8

ORGANIZING AND DESCRIBING DATA

PURPOSE OF UNIT

Pupils (and adults) are bombarded daily with a myriad of "facts and figures." It is essential that people be aware of some of the ways of organizing and describing these many facts so that the information presented can be more clearly understood.

This unit introduces the use of tables for presenting data in an organized manner and shows how this data can be pictured by graphs. It discusses how data can be described by numbers which represent the mean, median, or mode.

It is hoped that by the end of this unit the pupil will:

1. Recognize the need for organizing data and have some skill in making and reading tables.
2. Develop an understanding and some skill in "rounding off" numbers.
3. Have a clearer concept of what "graphing data" means.
4. Develop skill in reading, interpreting, and constructing simple line segment and bar graphs.
5. Be able to read and interpret circle graphs and to construct some of the more simple types of circle graphs.
6. Have been introduced to the idea of pictographs and their use and limitations.
7. Be aware of the concept of median, mean, and mode and be able to compute and identify these measures of central tendency.

MATHEMATICAL BACKGROUND

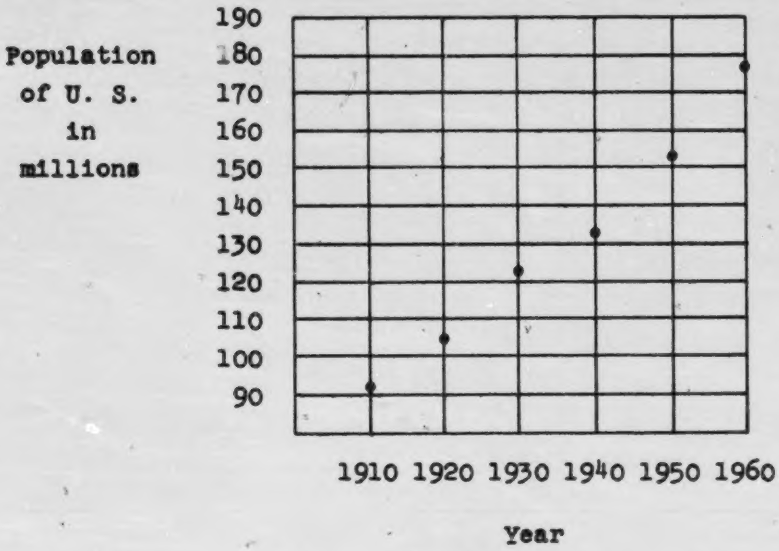
We live in a world of a bewildering and ever-increasing number of facts. An increasing proportion of this information is given in a numerical form. There are various ways to try to make this mass of data easily understood (the word "data" is the plural of the Latin word "datum" which means "fact"). A single number may be chosen which in some sense is an "average" of a large set of numbers. A number may be chosen which describes how the data varies from the "average." Various types of graphs may be drawn to illustrate the data. This chapter is an introduction to some of the ideas of organizing and describing data.

The first step in attempting to understand a large number of facts is to organize them in some type of table. In the table the facts are arranged in some order, and thus a table helps us to understand the information. Sometimes placing the data in groups will be an aid to understanding. For example, the teacher might get a clearer picture of the performance of a class on a test if she grouped all of the scores from 91 through 100 in one group, 81 through 90 in another group, etc. Grouping data in this fashion is not discussed in the pupil text.

Quite often we will represent the data by drawing a picture of it. Associated pairs of numbers are what we really expect a table to display. On a population table, we read a year and the population of that year. Hence, the ordered pair might be -- "1910, 92 million." The data could be pictured by using two perpendicular lines and labeling one line "years" and the other "population." A proper scale would have to be chosen to keep the graph on the sheet of paper. Then we could plot the pairs of points as in the earlier work on graphing in Chapter 5. When "time" (measured in hours, years, decades, for example) is one of the two sets of numbers, it is usually plotted on the horizontal axis.

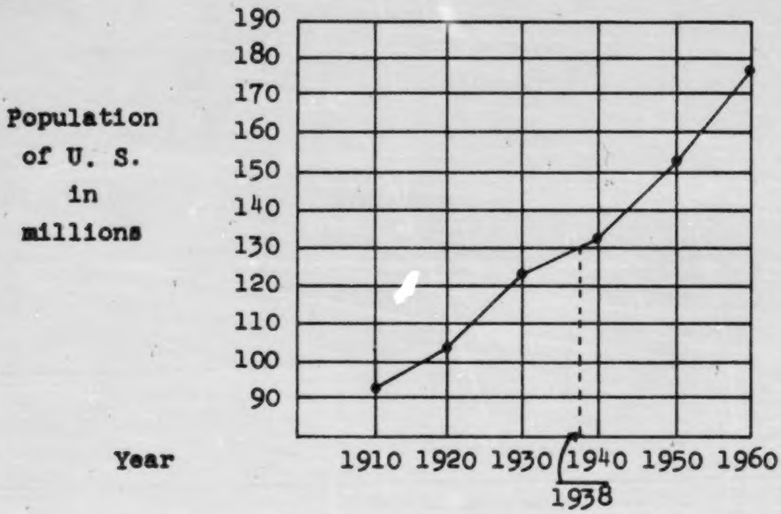
A graph might look like this if the data were the population of the United States:

GROWTH OF UNITED STATES POPULATION



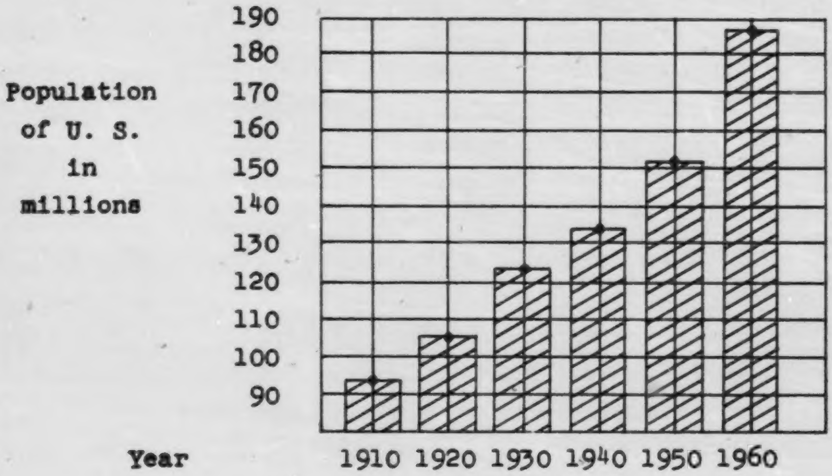
Sometimes some device is used to make the trend more apparent. One way to do this is to connect the points with line segments. A line segment graph should be drawn only when points between the endpoints of each line segment represent an estimate of some ordered pair of numbers that could be considered part of the data. The population graph shown here does not give us a record of the population in the year 1938, for example. But if we select a point on the line segment graph between the years 1930 and 1940 that would correspond to the year 1938 then the table would lead us to estimate the population at about 130 million. This is indicated in the graph. Thus we have made a tacit assumption that the population between 1930 and 1940 increased as shown by the graph. Of course, the population might have decreased between 1930 and 1935, then risen rapidly between 1935 and 1940. We do not know. In the absence of such information the line segments are drawn between points which represent known parts of the data. It may be said the line graph "carries the eye" in helping to read from the graph easily the information pictured. The graph obtained by connecting the points with line segments is called a line segment graph. The line segment graph of the population data plotted above would look like this:

GROWTH OF UNITED STATES POPULATION



Another way to make the trend easier to see is to draw bars from one of the axes to the points. As a bar graph, the population picture might look like this:

GROWTH OF UNITED STATES POPULATION

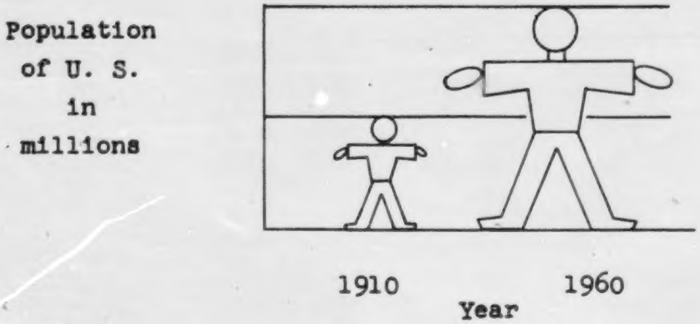


In a bar graph, the bars should be the same width and the space between the bars should be the same width but it is not necessary for the bars to be the same width as the spaces between them.

It is easier to compare the population of one year with that of another on a bar graph, but it is probably easier to visualize the change in population from the line segment graph. Some tables which can be illustrated by bar graphs cannot be illustrated by line graphs.

Many newspapers and magazines replace bar graphs by pictorial charts, or pictographs, where the bars are replaced by pictures of the objects. For example, using a picture of a man in the population chart, a comparison of the 1910 and 1960 population might look like this:

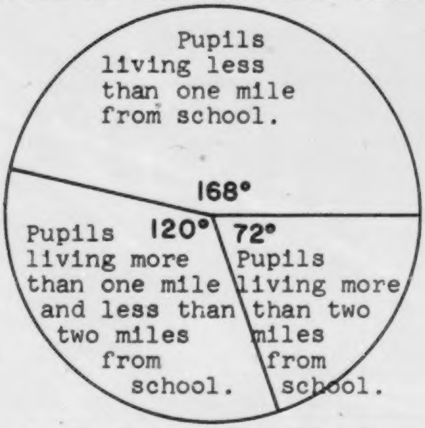
GROWTH OF UNITED STATES POPULATION



This picture is very misleading since, due to the three-dimensional suggestion of the pictures, the 1960 population appears to be much more than double the 1910 population. A fairer representation could be made by drawing a picture of a man for each 10 million people. One purpose of this unit is to teach the pupils to read graphs correctly and be wary of misleading information, such as might be given in the population pictograph. The ability to read graphs is more important for the pupils than the ability to make graphs, but the making of graphs will help the pupils to learn to read them.

When the data are concerned with the parts of a whole of something, then a circle graph may be used. By using the degree as the unit, 360 non-overlapping angles may be drawn with their vertices at the center of a circle. We can think, then, of a circular region separated into 360 regions, each shaped like a "slice of pie." Suppose we wished to make a circle graph to represent the distances the pupils in a particular class live from school. Of the 30 pupils in the class, 14 live less than one mile, 10 live more than one mile and less than two miles, and 6 live more than two miles from the school. Since 14 is $\frac{14}{30}$ of 30, the part of the interior of the circle that should be used to represent the 14 children is formed by an angle of 168 degrees ($\frac{14}{30} \times 360 = 168$). The angle will have its vertex at the center and radii will be parts of the rays. Hence the "slice of pie" in the interior of an angle will represent the portion of the class living less than one mile from school. We can perform a similar computation for the other two parts and then draw a circle graph:

NUMBER OF PUPILS AND DISTANCE THEY LIVE FROM SCHOOL



Many times it is desirable to have just one number to represent the data. If the number describes the tendency of the data to group about some "center," it is called a measure of central tendency, or an average. The three types of averages studied in this unit are the arithmetic mean, the median, and the mode. In this usage the word "arithmetic" is an adjective and its pronunciation should be that given by your classroom dictionary. When the teacher says the average grade on a test is 80, this conveys little information until the teacher states what type of average she is using.

The arithmetic mean of a set of numerical grades is obtained by adding the grades and dividing the sum by the number of grades. If there are some extreme grades in the set, then the arithmetic mean may not be a very good average to use. For example, if there are five boys in a class and if four boys made grades of 100 on a test and one boy made zero, then the arithmetic mean is 80. Commonly, the word "average" is used when "arithmetic mean" is intended.

The median of a set of numbers is a number such that half of the numbers in the set are greater than it and half are less. When the numbers are arranged in order of size, the median is the middle number. If there is an even number of numbers, then the median is the number half-way between the two "middle" numbers. For example, if the grades of six girls are 72, 78, 83, 85, 92, and 97, then the median is 84. Note that the median would still be 84 if the lowest grade were zero instead of 72. Many times the median is a better average to use than the arithmetic mean since extreme scores do not affect the median.

The third type of average is the mode, and it is defined as the number that occurs most frequently in the distribution. If the grades on a test are 72, 75, 76, 76, 76, 85, 92, 97, then the mode is 76. If the scores are 72, 75, 76, 76, 76, 78, 82, 82, 82, 85, 92, then there are two modes, 76 and 82. The distribution in this case is said to be bimodal.

If more men wear a size 8 shoe than any other size, the mode of these data is 8. The arithmetic mean might be $8\frac{7}{100}$ and a shoe manufacturer would be wise to use the mode rather than the arithmetic mean as a basis for a decision on the size of shoes to make.

TEACHING THE UNIT

The general plan of the unit is to introduce the main ideas through a section of Exploration. After the ideas of the Exploration have been developed under the teacher's guidance, these ideas are again reinforced through the use of appropriate Exercise Sets. Although the Exploration is teacher-directed, the pupils should be encouraged to participate actively.

Due to the many tables and a great amount of data used in the Exploration of each section, it is recommended that the pupils keep their books open during discussion. There are some parts of the Exploration that the pupils can study individually. The Exercise Sets should be done independently by each pupil. After individual study and completion of each Set of Exercises is accomplished, class discussion should further develop the concepts. The exercises serve two main purposes: to develop an idea, and to expand and fix an idea in mind. Encourage pupils to answer questions which other pupils raise about a particular exercise.

ORGANIZING DATA

Objectives: To develop the concept of data

To show how a table helps to picture organized data

To show that organization of data helps us to understand the data

Materials: paper, pencil, straightedge, chalk, chalkboard, yardsticks, ballots

Vocabulary: data, mammal, reptile, vary, tedious

Teaching Procedure:

In order to give this section more meaning it would be helpful to have 27 ballots prepared with the names of hamster, garter snake, turtle, lizard, and guinea pig written on them. The number of ballots should be in accordance with the number given in Table 1. Place the ballots on a table. Ask the children to answer question 1 of the Exploration. Do not arrange the ballots in rows or columns but just allow them to be a "hap-hazard" mass of information. This will "bring home" to the pupils the realization that data must be organized to be interpreted quickly.

The pupils need to keep their books open during the Exploration because of the large number of facts presented.

Some pupils may need help in arranging material in a table, especially if many facts are being presented. Some individual help may have to be given by the teacher in making a table to organize the information given, as in exercise 4. After the information is arranged in a table, the questions are easily answered.

The attention of the pupil should be directed to the Summary at the end of this first Exploration.

Chapter 8

ORGANIZING AND DESCRIBING DATA

INTRODUCTION

It would be a very tedious job to study the score card of every baseball game in which Mickey Mantle has played in order to determine how many times he has been at bat and how many times he has obtained hits. But we know at a glance how successful he is as a batter when we say his batting average this year is .326, if we know what "average" means, and whether .326 is "good."

There are many, many facts about today's world. Many of these facts are expressed in terms of numbers. There are facts about people's heights and weights, facts about baseball, facts about heights of buildings, facts radioed from satellites in outer space, facts about population, and many others.

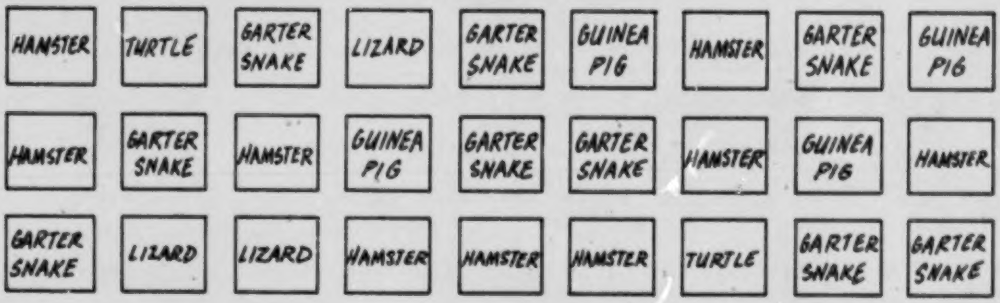
In this chapter you are going to study some ideas about organizing and describing data. You are going to learn that the word "average" has several meanings and that sometimes an "average" is very misleading. You are also going to learn various ways of presenting facts with graphs.

ORGANIZING DATA

Exploration

The teacher and the boys and girls in a sixth grade class wanted to buy a pet for their room. After some discussion, they decided to have each person write his choice on a slip of paper. They could choose from one of these: guinea pig, turtle, lizard, hamster, or garter snake.

When the slips were collected, they looked like this:


HAMSTER
TURTLE
GARTER SNAKE
LIZARD
GARTER SNAKE
GUINEA PIG
HAMSTER
GARTER SNAKE
GUINEA PIG
HAMSTER
GARTER SNAKE
HAMSTER
GUINEA PIG
GARTER SNAKE
GARTER SNAKE
HAMSTER
GUINEA PIG
HAMSTER
GARTER SNAKE
LIZARD
LIZARD
HAMSTER
HAMSTER
HAMSTER
TURTLE
GARTER SNAKE
GARTER SNAKE

1. How could the children decide which animal to buy? (By counting the vote Could they tell by just glancing at the votes or did they need to organize the information given by these votes? (They would need to organize the material.) Would the class need to vote again? (Only in case of a tie vote.)

When we think of arranging information in some way so that it can be more easily understood, we say we are organizing data. Data is another name for information or groups of facts.

2. A committee counted the votes for the classroom pet. Then they made a table showing the facts given by the votes.

It looked like this:

TABLE 1

Votes for Class Pet

Guinea Pig	4
Turtle	2
Lizard	3
Hamster	9
Garter Snake	9

A table always has a title to show what the facts are about.

a) Is it easier to tell by looking at this table or by looking at the slips of paper, how many votes each animal received? (It is easier to tell by looking at this table.)

The sixth graders decided to vote again to choose between a hamster and a garter snake. After the votes were counted, a table was made:

TABLE 2

Final Vote for Class Pet

Hamster	15
Garter Snake	12

b) Tell how a table helps you understand a set of facts. (It is easier to understand the facts if they are organized.)

3. George, Harry, Jim, and John each have a newspaper stand. The number of papers they sold one week was: Monday - George 3, Harry 28, Jim 15, John 36; Tuesday - George 10, Harry 29, Jim 21, John 38; Wednesday - George 18, Harry 29, Jim 47, John 38; Thursday - George 30, Harry 30, Jim 47, John 40; Friday - George 52, Harry 32, Jim 47, John 42; Saturday - George 70, Harry 60, Jim 50, John 44; Sunday - George 98, Harry 75, Jim 54, John 45.

Without organizing the data, answer these questions:

a) On what day did Jim sell as many papers as George and Harry together? (Wednesday)

b) Which boy sold the same number of papers three days in a row? (Jim)

c) Which boy sold more papers each day than he sold the day before? (George)

d) Which boys sold more papers, or at least as many as the day before, each day through the week? (all)

e) Which boy sold more than three times as many papers one day as he did the day before? (George)

f) Which boy's sales for each day of the entire week did not vary over 10 papers? (John)

Now look at the data organized in this table. Does the table make it easier to answer the questions?

TABLE 3

Newspaper Sales for One Week

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
George	3	10	18	30	52	70	98
Harry	28	29	29	30	32	60	75
Jim	15	21	47	47	47	50	54
John	36	38	38	40	42	44	45

P497

4. Eight Girl Scout troops in Centerville sold boxes of cookies. There were three kinds of cookies, one kind to a box. There were boxes of mint, chocolate, and vanilla cookies. The sales, in boxes, of each troop were: Troop 1 - mint 48, chocolate 63, vanilla 35; Troop 2 - mint 34, chocolate 27, vanilla 30; Troop 3 - mint 72, chocolate 51, vanilla 40; Troop 4 - mint 25, chocolate 14, vanilla 12; Troop 5 - mint 75, chocolate 39, vanilla 51; Troop 6 - mint 51, chocolate 62, vanilla 37; Troop 7 - mint 132, chocolate 98, vanilla 99; Troop 8 - mint 82, chocolate 98, vanilla 76. Make a table to show the cookie sales. Use the table to help you answer these questions: (See T.C. for the table.)

a) Which troop sold the most cookies? (Troop 7) (Would it help if you showed the total sales in the table?)

b) How many more boxes of chocolate cookies were sold than vanilla? (72) (Would it help to show the totals of each kind of cookie sold?) (yes)

Exercises a) and b) show that it sometimes helps to total some of the data given in a table and to show this total in the table.

c) Which troop sold the most boxes of chocolate cookies? (Troops 7 and 8) the least? (Troop 4)

d) Look at the part of the table which shows the number of boxes of mint cookies sold by each troop. Troop 7 sold one less box of mint cookies than the sales of two other troops combined. Name these two troops. (Troops 6 and 8)

e) Make up some questions of your own that can be answered by studying the table. (Answers will vary.)

Answer to exercise 4, Pupil's Text.

TABLE 3

Girl Scout Cookie Sales

Troop	Mint	Chocolate	Vanilla	Total
1	48	63	35	146
2	34	27	30	91
3	72	51	40	163
4	25	14	12	51
5	75	39	51	165
6	51	62	37	150
7	132	98	99	329
8	82	98	76	256
TOTAL	519	452	380	1351

SUMMARY

We have learned:

1. Data is another name for information or groups of facts.

2. We organize data to make it more easily understood.

3. We can organize data by using tables.

4. Tables have titles to help us understand the data.

5. It is sometimes helpful to add the numbers given in tables and show the sums in the table.

Exercise Set 1

1. Linda and Perry's arithmetic scores for two weeks are given below. The scores tell the number of correct answers.

Linda: Monday - 18, Tuesday - 54, Wednesday - 12,
Thursday - 44, Friday - 96; Monday - 7, Tuesday - 75,
Wednesday - 20, Thursday - 35, Friday - 72.

Perry: Monday - 20, Tuesday - 54, Wednesday - 12,
Thursday - 43, Friday - 97; Monday - 6, Tuesday - 77,
Wednesday - 19, Thursday - 36, Friday - 70.

Make a table to show these facts. (See T.C.)

Which of these sixth-graders had the higher total score. (Perry)

2. Six boys are going on a hike. They are going to take the following things with them:

Bill - canteen, ax, cookies

Chuck - sandwiches, canteen, cookies

Lee - knife, compass, wieners

Dick - beans, canteen, first aid kit

Jack - matches, sandwiches, wieners

Sam - buns, beans, canteen

Make a table to show the number of boys who are taking canteens, cookies, etc. (See T.C.)

a) How many different items are the boys taking? (11)

b) How many boys are taking wieners? (2) buns? (1) beans? (2) canteens? (4)

3. The "home states" of the Presidents of the United States are: Washington - Virginia; J. Adams, Massachusetts; Jefferson - Virginia; Madison - Virginia; Monroe - Virginia; J. Q. Adams - Massachusetts; Jackson - South Carolina; Van Buren - New York; W. H. Harrison - Virginia; Tyler - Virginia; Polk - North Carolina; Taylor - Virginia; Fillmore-New York; Pierce - New Hampshire; Buchanan - Pennsylvania; Lincoln - Kentucky; Johnson - North Carolina; Grant - Ohio; Hayes - Ohio; Garfield - Ohio; Arthur - Vermont; Cleveland - New Jersey; Harrison - Ohio; McKinley - Ohio; T. Roosevelt - New York; Taft - Ohio; Wilson - Virginia; Harding - Ohio; Coolidge - Vermont; Hoover - Iowa; F. D. Roosevelt - New York; Truman - Missouri; Eisenhower - Texas; Kennedy - Massachusetts.

Make a table to show the number of presidents from each state.

a) How many presidents were from the six New England states? (6)

b) How many presidents did Ohio and Virginia together furnish? (15)

c) How many presidents were from west of the Mississippi River? (3)

d) What 3 states furnished over half of the presidents? (Virginia, Ohio, New York)

e) Name the states in which more than two presidents were born. (Mass., Virginia, New York, Ohio)

Answer to exercise 1, Set 1

TABLE 1

Arithmetic Scores of Two Sixth-Graders

	Monday	Tuesday	Wednesday	Thursday	Friday	Total
LINDA						
First Week	18	54	12	44	96	
Second Week	7	75	20	35	72	
TOTAL	25	129	32	79	168	433
PERRY						
First Week	20	54	12	43	97	
Second Week	6	77	19	36	70	
TOTAL	26	131	31	79	167	434

Answer to exercise 2, Set 1

TABLE 2

Things Taken by Six Boys on a Hike

	Canteen	Ax	Cookies	Sandwiches	Knife	Compass	Wieners	Beans	Buns	Mattresses	First Aid Kit
Bill	✓	✓	✓								
Chuck	✓		✓	✓							
Lee					✓	✓	✓				
Dick	✓							✓			✓
Jack				✓			✓			✓	
Sam	✓							✓	✓		
TOTAL	4	1	2	2	1	1	2	2	1	1	1

254

Answer to exercise 3, Set 1

Home States of Presidents

Iowa	1	Hoover
Kentucky	1	Lincoln
Massachusetts	3	J. Adams, J. Q. Adams, Kennedy
Missouri	1	Truman
New Hampshire	1	Pierce
New Jersey	1	Cleveland
New York	4	Van Buren, Fillmore, T. Roosevelt, F. D. Roosevelt
North Carolina	2	Polk, Johnson
Ohio	7	Grant, Hayes, Garfield, Harrison, McKinley, Taft, Harding
Pennsylvania	1	Buchanan
South Carolina	1	Jackson
Texas	1	Eisenhower
Vermont	2	Arthur, Coolidge
Virginia	8	Washington, Madison, Jefferson, Monroe, Harrison, Tyler, Taylor, Wilson

"ROUNDING OFF NUMBERS"

Objectives: To show how numbers are approximated by "rounding off"

To give a working process for rounding off numbers to the nearest convenient unit for the purpose at hand

Materials: Chalk, chalkboard, straightedge, pencil, paper

Vocabulary: Estimate, approximations, rounded number, odometer

Teaching Procedure:

The phrase "Rounding Off Numbers" requires some explanation. It is used to convey the meaning indicated in the following examples.

"Rounding Off 74" to the nearest multiple of 10 means to replace 74 by that multiple of 10 which would label a point on the number line that is the smallest distance from the point that labels 74. Of all the multiples of ten the two nearest to 74 are 70 and 80. Since 80 - 74 = 6 and 74 - 70 = 4, the multiple of ten nearest to 74 is 70. Consequently "rounding off 74" to the nearest multiple of 10 simply means to replace 74 by 70. Rounding off 76 to the nearest multiple of 10 would require replacing 76 by 80; 21 by 20; 39 by 40; 999 by 1000; 1004 by 1000. By agreement, if a number is halfway between two multiples of ten it is "rounded off" to the larger of the two multiples of ten. Thus, 75 is "rounded off" to 80; 1095 to 1100; 705 to 710, etc. In a similar manner numbers are "rounded off" to the nearest multiple of 5, or of 100, or of 1000, etc., as described in the pupil's book.

With this accepted meaning of "rounding off numbers" the phrase will be used hereafter without the quotation marks. A suitable explanation of the meaning of the phrase should be presented to the pupils.

The opening paragraphs are intended to alert the pupil to the need for being precise in using some numbers and the need for estimating in using other numbers. Many other examples may be given and it would be well to permit the pupils to indulge in some imaginative discussion at this point on when to be exact and when to "round off."

The number line is a very helpful device in giving the child a mental picture of the process of rounding off a number. Simple illustrations are given at first since some members of the class may not have had any instruction in this area.

The use of a comparable number line by the teacher at the chalkboard will greatly speed mastery of the process. During this Exploration, the class and teacher, working together with books open, will provide the most profitable use of time.

As each new step is presented, exercises are given for the class to work.

As the lesson is developed, the teacher should encourage pupils to recognize patterns. As soon as some pupils see the whole picture, they should be encouraged to explain their findings to others.

Ask the pupils to give numbers which they would like to have rounded off. Ask them what multiples they want to use for the rounding off. Encourage examples which are not already in the text.

This Exploration and Set of Exercises has been included here to prepare the pupil with the needed skill in making graphs. Large numbers need to be rounded off for conveninece in graphing.

The set of exercises on rounding off numbers following this Exploration should be done independently by the pupils.

ROUNDING NUMBERS:

There are times when we need to know the exact number of objects in a set. At other times we need only to know "about how many." The swimming instructor finds it necessary to know the exact number of children that go into the pool. The same number of children should come out, at the end of the period, as went in it at the beginning of the period. It is not enough for him to know only "about how many" children there are.

There are other numbers that need to be known exactly. Some numbers such as your telephone number, your locker number, and your house number need to be stated exactly.

There are many times when we can use a number which tells "about how many." Most measurements can be made only with a certain degree of accuracy. These "about how many" numbers are sometimes called estimates, approximations, or rounded numbers.

We use rounded numbers at times when we need only an estimate. The police estimate a crowd of people at a parade at 200,000. The traffic officer may estimate the speed of the passing auto at 60 miles per hour. These estimates are given in round numbers. What other examples can you give of numbers which tell "about how many?" (Answers will vary.)

Sometimes we replace a count or measurement of something by a less accurate one. It might not be important to use the number of people in the United States as 179,323,175. This number was given by the census of 1960. Do you think this number is exactly correct? Would we be correct if we said there were nearly 180,000,000 people in the United States in 1960?

Suppose we wanted to know what part of the people of the United States lived in the state of New York in 1960. The census reported that 14,759,429 people lived in New York. Do you think this number is exactly correct? Would 15,000,000 be nearly correct? If we used one of these fractions to tell us what part of the U. S. people lived in New York, which one would be easier to write in a simpler form:

$$\frac{15{,}000{,}000}{180{,}000{,}000} \quad \text{or} \quad \frac{14{,}759{,}429}{179{,}323{,}175}$$

There are many other times when we should use rounded-off numbers in place of exact measures. Think of this example. Suppose you are making a trip in your car. On the first part of the trip you read from the odometer that you travel 121.7 miles. On the second part of the trip you forget to read the odometer. But the map gives the mileage between 95 and 105. Now what numbers can you add to find how far you traveled on the trip? Should you use 121.7 miles for the first part of the trip? What mileage would you use for the last part of the trip? If you round off 121.7 to the nearest multiple of ten the number would be 120. The multiple of ten between 95 and 105 is 100. If you said that you traveled 120 + 100 or 220 miles on the entire trip, would 220 be about right?

There are several ways of "rounding off" numbers. For our work here, we will use a very simple way, as is shown on the following page.

Below is a number line:

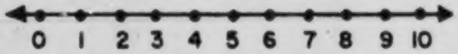


Figure 1

Let us use only the segment from 0 to 10. What is the middle point of this segment of the number line? (5)

We sometimes round off numbers to the nearest multiple of 10. If we round off the number 7 to the nearest multiple of 10, we say it is rounded to 1 ten. The number 4 is rounded to 0 tens.

Round off 6 to the nearest multiple of 10. (10)

What happens if we round off 2 and 3 to the nearest multiple of 10? (2 and 3 round off to zero tens or 0.)

In rounding off to the nearest multiple of ten, what do you suppose happens to any number which is to the left of point 5 on the number line? (It is rounded to zero tens.)

What would a number to the right of five become? (one ten)

How do we round off the number five to the nearest multiple of ten? (We shall round off 5 to 1 ten as a rounded number.)
We call five, one ten as a rounded number.)

Below is another section of a number line:

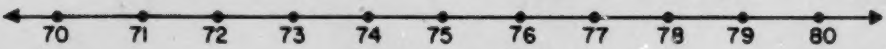


Figure 2

How would we round off the numbers 70, 71, 72, 73, 74 to the nearest multiple of ten? (to 70)

How would we round off the numbers 75, 76, 77, 78, 79 to the nearest multiple of ten? (to 80)

Round these numbers to the nearest multiple of ten.

a) 147 (150)

b) 132 (130)

c) 984 (980)

d) 565 (570)

e) 2,168 (2,170)

f) 995 (1,000)

In rounding off to the nearest multiple of ten, what do we do when the number 5 is in the one's place? (We replace the number by the next multiple of 10 which is greater than the given number; e.g., 5 would be rounded to 10, 15 rounded to 20; 135 rounded to 140, etc.)

If we round off a number to the nearest multiple of ten, we replace the number by the nearest multiple of 10.

We round off 16 to 20, 37 to 40, 89 to 90, 81 to 80, 176 to 180.

If the number to be rounded off to the nearest multiple of 10 is halfway between two multiples of 10, we replace the number by the next larger multiple of 10.

We round off 15 to 20, 25 to 30, 115 to 120, 905 to 910, etc.

If we round off a number to the nearest multiple of one hundred, we replace it by the nearest multiple of 100.

We round off 46 to 0, 89 to 100, 176 to 200, 341 to 300, etc., if we are rounding off to the nearest multiple of 100.

If the number to be rounded off to the nearest multiple of 100 is halfway between two multiples of 100 we replace the number by the next larger multiple of 100.

We round off 50 to 100, 150 to 200, 250 to 300, etc., if they are to be rounded off to the nearest multiple of 100.

Now do you see how to round off a number to the nearest multiple of 1000? of 10,000, etc.?

In the section of a number line below we have the numbers by tens from 100 to 200.

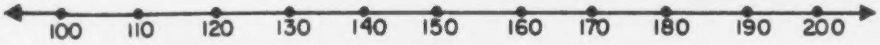


Figure 3

What is the middle number between the numbers 100 and 200? (150)

Round off 140 to the nearest multiple of one hundred. (100)

Round off 180 to the nearest multiple of one hundred. (200)

If the number in the tens' place is less than five as in 110, 120, 130, 140, what is the nearest hundred? (100)

If the tens' place digit is 5 or greater, what is the nearest multiple of one hundred on the above number line? (200)

Do you suppose this holds true for rounding off all numbers to the nearest multiple of hundred? (yes)

Round off these numbers to the nearest multiple of one hundred.

a) 92 (100)
b) 550 (600)
c) 602 (600)
d) 3,849 (3,800)
e) 1,841,731 (1,841,700)

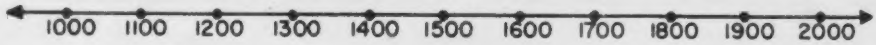


Figure 4

Dick had the above diagram to look at. He was asked to round off the following numbers to the nearest multiples of one thousand: 1400, 1700, 1500. He answered that 1400 rounds off to 1000, 1700 and 1500 round off to 2000. Was he right? (yes) Why? (Because 1,400 is closer to 1,000 than it is to 2,000; 1700 is closer to 2,000 than it is to 1,000; 1500 has a 5 in the hundreds place so we round it up to 2,000.)

If the number in the hundreds place is 5 or more, in rounding to the nearest multiple of one thousand, what do you do with the number in the thousands' place? (add 1 to the number in the thousands place.)

If the number in the hundreds' place is less than 5, what happens? (The number in the thousands' place remains the same.)

Round off these numbers to the nearest multiple of one thousand:

a) 455 (0)
b) 1,100 (1,000)
c) 8,545 (9,000)
d) 110,905 (111,000)
e) 9,998 (10,000)
f) 67,496 (67,000)

Do you see a pattern in this?

If the number in the thousands' place is 5 or more would you increase the digit in the 10,000 place by one? (yes) What would you do if the number in the thousands' place is less than 5? (Do not change the digit in the ten thousands place.)

Round these numbers to the nearest multiple of 10,000:

a) 15,000 (20,000)
b) 11,111 (10,000)
c) 9,200 (10,000)
d) 184,000 (180,000)
e) 7,775,600 (7,780,000)

If you use place value of the digits in a numeral, it may help you in rounding off a number. Look at these examples.

a) Round off 3,495,000 to the nearest multiple of 1,000,000. Think of the number as 3 million + 495,000. We want to know if it should be rounded off to 3 million or 4 million. We look at 495,000 and see that it is less than one-half of 1,000,000. This tells us to round off the number to 3 million, 3,000,000.

b) Round off 7,775,600 to the nearest multiple of 100,000. Think of the number as 77 hundred thousands + 75,600. We want to know if it should be rounded off to 78 hundred thousands or to 77 hundred thousands. We look at 75,600 and see that it is more than one-half of 100,000. This tells us to round off the number to 78 hundred thousands, 7800,000 or 7,800,000.

These examples help us see that just one digit is important in rounding off a number.

In rounding off to the nearest multiple of one million, the hundred thousands' digit is important. If it is 5 or greater than 5, then the millions' digit is increased by 1. But if the hundred thousands' digit is less than 5, then the millions' digit is not changed.

In rounding off to the nearest multiple of one hundred thousand, the ten thousands' digit is important. If it is 5 or greater than 5, then the hundred thousands' digit is increased by 1. But if the ten thousands' digit is less than 5, then the hundred thousands' digit is not changed.

Does this help to see that in rounding off to the nearest 10,000 that the thousands' digit is important? Which digit is important in rounding off to the nearest 1000? (the hundreds' digit) the nearest 100? (the tens' digit) the nearest 10? (the units' digit)

Copy each of these numbers. Draw a circle around the important digit. Then round off to the nearest multiple of one hundred thousand:

a) 7④9,000 (700,000)

b) 7⑤0,000 (800,000)

Round off to the nearest multiple of one million after circling the important digit.

c) 3,④95,000 (3,000,000)

d) 3,⑥75,112 (4,000,000)

e) 9,⑤00,000 (10,000,000)

Now suppose you wished to round off to the nearest multiple of 5 or 50 or 500, how would you go about doing it?

We will use a number line again to help us in our thinking.

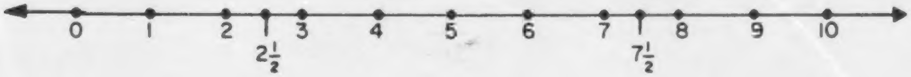


Figure 5

What is the mid-point between 0 and 5? (Between 2 and 3, point $2\frac{1}{2}$ or $\frac{5}{2}$.)

What is the mid-point between 5 and 10? ($7\frac{1}{2}$ or $\frac{15}{2}$)

If we are rounding off to the nearest multiple of five, which numbers on the number line would be rounded to five? (all numbers from $2\frac{1}{2}$ to $7\frac{1}{2}$)

Would $2\frac{1}{2}$ be rounded to 5? (yes) Would $7\frac{1}{2}$ be rounded to 5? (no)

What would $7\frac{1}{2}$ be rounded to in groups of five? (In groups of five or ten)

What would 1 and 2 be rounded to in groups of five? (zero)

Round off to the nearest multiple of 5:

a)	4 (5)	c)	7 (5)	e)	12 (10)
b)	9 (10)	d)	16 (15)	f)	342 (340)

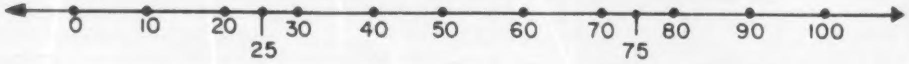


Figure 6

On the number line above what point is mid-way between 0 and 50? (25)

What point is mid-way between 50 and 100? (75)

What numbers could be rounded off to one multiple of 50? Numbers from 25 to, but not including, 75)

Would 25 be rounded off to 1 multiple of 50? (yes)

If 75 were rounded off to the nearest multiple of 50, what would be the rounded number? (100)

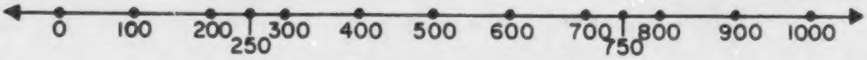


Figure 7

On this number line what point is midway between 0 and 500? (250) Between 500 and 1000? (750) What numbers can be rounded off to 500? Numbers from 250 to, but not including, 750) Would 250 be rounded off to 500? (yes) Round off 750 to the nearest multiple of 500. (1,000)

What is the pattern going to be if we round off to the nearest multiple of five, 50, 500, or 500,000? In rounding off to the nearest multiple of 500, the important digits in the following numbers are circled: 1325; 23,726; 91,304; 126,253. The rounded numbers for these are: 1300; 23,500; 91,500; 126,500.

Round off these numbers to the nearest multiple of 500,000.

a) 140,000 (0)
b) 300,000 (500,000)
c) 1,600,000 (1,500,000)
d) 250,000 (500,000)
e) 750,000 (1,000,000)

In locating points on a graph it would not be reasonable to graph ordered pairs expressed by exact numbers when the numbers are large. It would be very difficult to choose a scale in which we could show 40,000, 50,000, and 49,876 on a number line and show a difference between the last two numbers that we could use. Therefore, in graphing data using very large numbers we estimate or round off the numbers to fit the most convenient scale.

Exercise Set 2

1. Round each of these numbers to the nearest multiple of 10.

a) 14 (10) c) 20 (20) e) 199 (200)

b) 15 (20) d) 555 (560) f) 3999 (4,000)

2. Round each of these numbers to the nearest multiple of one hundred.

a) 65 (100) b) 102 (100) c) 149 (100) d) 851 (900) e) 990 (1,000)

3. Round each of these numbers to the nearest multiple of one thousand.

a) 8,052 (8,000) c) 9,499 (9,000) e) 10,050 (10,000)

b) 3,716 (4,000) d) 9,500 (10,000)

4. Write each of these numbers rounded to the nearest multiple of ten thousand, the nearest multiple of one hundred thousand, and the nearest multiple of one million. (For results, see next page of T.C.)

a) 55,600 c) 1,098,760 e) 9,615,847,100

b) 585,500 d) 30,500,001 f) 105,105,105

5. Round the following numbers to the nearest multiple of five hundred, the nearest multiple of five thousand, the nearest multiple of five hundred thousand.

a) 2649
(2,500
5000
0)

b) 864,492
(864,500
865,000
1,000,000)

c) 7,048,501
(7,048,500
7,050,000
7,000,000)

Answer to exercise 4, Set 2

	ten thousand	hundred thousand	million
a) 55,600	60,000	100,000	0
b) 585,500	590,000	600,000	1,000,000
c) 1,098,760	1,100,000	1,100,000	1,000,000
d) 30,500,001	30,500,000	30,500,000	31,000,000
e) 9,615,847,100	9,615,850,000	9,615,800,000	9,616,000,000
f) 105,105,105	105,110,000	105,100,000	105,000,000

6. The population in 1960 of some of the states is given below. Their rank, according to population, is given for 1950 and for 1960. Make a table to show this information. Use four columns: State, 1960 Population, 1960 Rank, 1950 Rank. List the state with the largest population first and list the others in order of decreasing population.

(For results, see next page of T.C.)

Alabama	3,266,740;	1950 rank 17;	1960 rank 19
Arizona	1,302,161;	1950 rank 38;	1960 rank 35
California	15,717,204;	1950 rank 2;	1960 rank 2
Connecticut	2,535,234;	1950 rank 28;	1960 rank 25
Florida	4,951,560;	1950 rank 20;	1960 rank 10
Hawaii	632,772;	1950 rank 46;	1960 rank 44
Illinois	10,081,158;	1950 rank 4;	1960 rank 4
Iowa	2,757,537;	1950 rank 22;	1960 rank 24
Kentucky	3,038,156;	1950 rank 19;	1960 rank 22
Maine	969,265;	1950 rank 35;	1960 rank 36
Massachusetts	5,148,578;	1950 rank 9;	1960 rank 9
Minnesota	3,413,864;	1950 rank 18;	1960 rank 18
Missouri	4,319,813;	1950 rank 11;	1960 rank 13
Nebraska	1,411,330;	1950 rank 33;	1960 rank 34
New Hampshire	606,921;	1950 rank 45;	1960 rank 46
New Mexico	951,023;	1950 rank 40;	1960 rank 37
New York	16,782,304;	1950 rank 1;	1960 rank 1
North Carolina	4,556,155;	1950 rank 10;	1960 rank 12
Ohio	9,706,397;	1950 rank 5;	1960 rank 5
Pennsylvania	11,319,366;	1950 rank 3;	1960 rank 3
Texas	9,579,677;	1950 rank 6;	1960 rank 6
Virginia	3,966,949;	1950 rank 15;	1960 rank 14
Washington	2,853,214;	1950 rank 23;	1960 rank 23
Wyoming	330,066;	1950 rank 49;	1960 rank 49

Answers to exercises 6 and 7, Set 2

State	1960 Population	1950 Rank	1960 Rank	Population to the Nearest multiple of 500,000
New York	16,782,304	1	1	17,000,000
California	15,717,204	2	2	15,500,000
Pennsylvania	11,319,366	3	3	11,500,000
Illinois	10,081,158	4	4	10,000,000
Ohio	9,706,397	5	5	9,500,000
Texas	9,579,677	6	6	9,500,000
Massachusetts	5,148,578	9	9	5,000,000
Florida	4,951,560	20	10	5,000,000
North Carolina	4,556,155	10	12	4,500,000
Missouri	4,319,813	11	13	4,500,000
Virginia	3,966,949	15	14	4,000,000
Minnesota	3,413,864	18	18	3,500,000
Alabama	3,266,740	17	19	3,500,000
Kentucky	3,038,156	19	22	3,000,000
Washington	2,853,214	23	23	3,000,000
Iowa	2,757,537	22	24	3,000,000
Connecticut	2,535,234	28	25	2,500,000
Nebraska	1,411,330	33	34	1,500,000
Arizona	1,302,161	38	35	1,500,000
Maine	969,265	35	36	1,000,000
New Mexico	951,023	40	37	1,000,000
Hawaii	632,772	46	44	500,000
New Hampshire	606,921	45	46	500,000
Wyoming	330,066	49	49	500,000

7. Round to the nearest multiple of 500,000 the population given in exercise 6. (For results, see previous page of T.C.)

   a) What three states have rounded populations of one-half million? (Hawaii, New Hampshire, and Wyoming)

   b) What three states have rounded populations of three million? (Iowa, Kentucky, and Washington)

   c) Which state has a rounded population of $2\frac{1}{2}$ million? (Connecticut)

   d) In rounding to the nearest multiple of 500,000, which states would appear to have larger populations than shown by the numbers in exercise 6? Which seem to have smaller populations?

Those appearing to have larger population:

Alabama
Arizona
Florida
Iowa
Maine
Minnesota
Missouri
Nebraska
New Mexico
New York
Ohio
Pennsylvania
Virginia
Washington
Wyoming

States appearing to have smaller populations:

California
Connecticut
Hawaii
Illinois
Kentucky
Massachusetts
New Hampshire
North Carolina
Texas

BRAINTWISTERS

1. Make a table of the population of the fifty states, using the final 1960 census figures. Arrange in order of population as you did in the preceding exercise 6.

2. Obtain data of interest to you. Show how it can be placed in a table to make it easy to understand.

GRAPHS OF DATA

Objectives: To introduce the idea that data can be represented by a graph

To build a foundation on which broken-line, bar, circle, and pictographs can be understood

Vocabulary: Horizontal reference line, vertical reference line

Materials: Chalkboard, chalk, straightedge, paper, pencil, graph paper

Suggested Teaching Procedure:

Follow the Exploration as given in the pupils' book. The pupils will have their books open.

The unit on Coordinates serves as a good background for this present work. Now the ordered pair is not two numbers but is, for example, "hamster and 9 votes." This ordered pair can be plotted on graph paper on which one axis (called a reference line) refers to Pets and the other axis to Number of Votes.

If you draw on the chalkboard the two graphs of Vote for Class Pet from pupils' book, you will help pupils understand more clearly how you "plot the ordered pairs." Many sixth-graders will have drawn graphs previously so your job will not be a difficult one on this particular data on the Vote for Class Pet.

However, children have difficulty in determining scales to be used in making graphs and they will need assistance on this. Example 2 of the Exploration is designed to give an introduction to it. You will need to carry this further. The first thing to do is to determine the number of units available for the graph on the graph paper. (The use of graph paper is advised for exercises in this unit.)

Many times the scale can be determined by just looking at the data to be shown by graph. If the numbers are large, there are two ways in which you might want the children to proceed.

One way is to divide the largest number of units (spaces or lines) available on the graph paper. The quotient will indicate the number that each unit is to represent. This quotient is rounded to a higher, convenient number. For example, if there are 18 units available on the graph paper and the largest number to be graphed is 8,240, each unit might represent 500. ($8,240 \div 18 = 457\frac{14}{18}$.) This method is used if the smallest number to be graphed is near zero; that is, if the numbers range from near zero to 8,240.

A second way to proceed would be to obtain the difference between the largest number and the smallest number to be represented on the graph. Then divide this difference by the number of units available on the graph paper. Round this quotient to a larger, more convenient number to find the number each unit might represent. Your graph then would NOT start at zero but would start at the next representative number below the lowest number to be graphed. For example, suppose the largest number we need to graph is 294,779 and the smallest is 243,684. The difference between these two numbers is 51,095. Assume we have 30 units available on the reference line, so we'll divide 51,095 by 30. This gives us 1703-5/30. We can let each of the units represent 2,000 and start our graph at 242,000. A shorter way to determine a suitable scale is to round 293,779 up to 295,000 and round 243,684 down to 240,000. The difference is 55,000 and we can see that with 30 units, each unit could represent 2,000 on this graph. Some children will be able to do this rounding of the numbers and choosing an approximate scale mentally.

On the graphs of Vote for Class Pet some children may want to connect points with line segments. Exercise 5 in the next Exploration explains why this should not be done with this particular data. The paragraph which precedes the population graph in the Mathematical Background explains the reasons for this in more detail. The only time when line segments may be drawn is when each point of the line segment would represent an estimate of some ordered pair of numbers that could be considered as part of the data.

Exploration

In the chapter on Coordinates you learned that you could draw a picture of a set of ordered pairs of numbers by associating each ordered pair with a point of the plane. The point was called the "graph" of the ordered pair. The tables in Chapter 8 contain examples of ordered pairs, although both members of the pair are not necessarily numbers. For example, in the vote on a class pet, we had the ordered pairs (hamster, 9 votes), (guinea pig, 4 votes), etc. Thus we should be able to set up some type of a coordinate system and locate points to correspond to our data. These points are called a graph of the data.

In graphing data, we usually use only one quadrant and the axes are called reference lines. It is not necessary to use the same scale on each reference line. For example, one reference line might represent years and the other reference line might represent population. Your reference line for years might only have to indicate numbers to a little beyond 1960, but your reference line for population might need to indicate numbers as high as 2,905,600,000. A graph has a title which tells what the graph shows. Graph paper is often used for constructing graphs.

1. The results of the vote for a class pet may be shown on a graph. But, instead of numbers, names of the pets would be listed on the vertical line. Five evenly-spaced points on the line are chosen and each one labeled with the name of one of the pets. It is easy to see that we need to use only 5 spaces because there are just 5 pets. If we wanted to have a graph that was "spaced out" more, we could have skipped one or two lines between the names of the pets. The pets' names would then be evenly spaced but would be farther apart. The largest vote for any pet was 9 so we choose 9 evenly-spaced points on the horizontal reference line and label this line "Number of Votes."

2. Let's suppose that all the children in the school were voting for a pet. Then one animal might get as many as 360 votes. We do not have 360 of the evenly-spaced vertical lines. Therefore, we cannot let each line stand for just one vote. We would need to let each line stand for several votes.

If we chose to let each line stand for 5 votes, then we would need 72 lines because there are 72 fives in 360. Are there 72 vertical lines on your graph paper? Probably not! Then let's let each line represent 25 votes, for example. We would need 15 lines because 14 twenty-fives would show 350 votes and we want to show more votes than that.

a) If we let each line stand for 50 votes, how many lines would we need? (8)

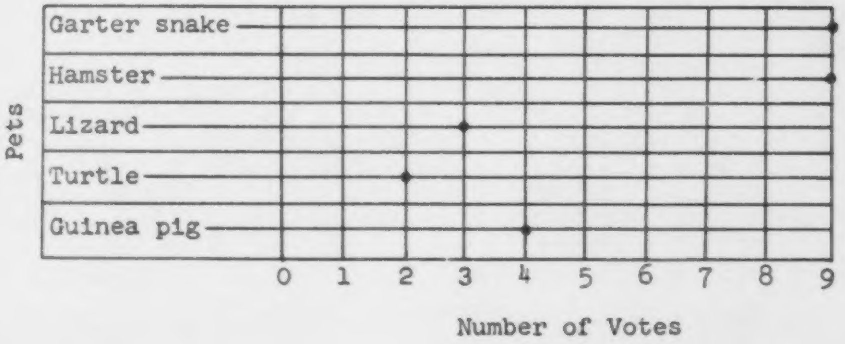
b) If we let each line stand for 10 votes, how many lines would we need? (36)

Before we can draw a graph, we must look at the facts we want to graph. We will need to decide how to show these facts on a graph. We may need to let the lines on the graph paper stand for more than just one fact, as we explained above.

Now locate the points corresponding to the vote for each pet. Our graph might look like this:

Graph of

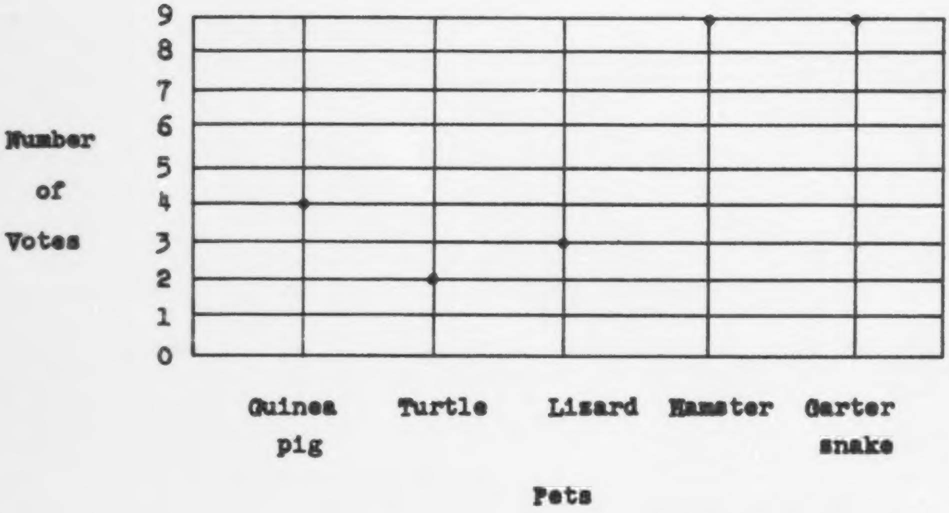
Vote for Class Pet



3. Could we have used the vertical axis to show the number of votes? Our graph then might look something like this:

Graph of

Vote for Class Pet



4. Does this graph tell the same story? (yes) From the graph, tell which pets received 0 votes, (none) 4 votes, (Guinea Pig) 9 votes. (Hamster and garter snake)

5. In some graphs the points are connected by line segments. These line segments show approximately where other points would be if additional data were obtained. In this graph, line segments would have no meaning and should not be drawn. For example, there isn't any animal halfway between guinea pig and turtle.

LINE SEGMENT GRAPHS

Objective: To develop skill in understanding and constructing line segment graphs and to learn when it is appropriate to represent data with this kind of a graph

Vocabulary: Line segment graph

Materials: Chalkboard, chalk, straightedge, paper, pencil, graph paper

Suggested Teaching Procedure:

Follow the Exploration as given in the pupil text. Pupils will have their books open.

Draw on the chalkboard a graph of the population data while each pupil draws one on his paper. It would be well to develop the following ideas concerning graph construction before the pupils start to work:

1. Read the data carefully.
2. Plan before making any marks on the paper. Consider, for example, what scales will be needed for the graph, where the title shall be placed and how much room will it take, and how each reference line shall be labeled. Be sure that the graphs are not too small.
3. Use manuscript writing.
4. Use a straightedge when drawing lines so they will be straight.
5. Graphs should be constructed so that they are "read" from left to right.

It is not necessary that the same scale be used for both reference lines.

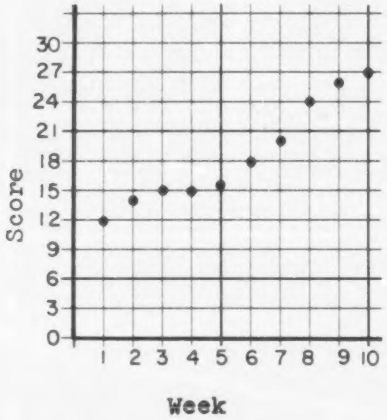
In example 1 of the Exploration on Line-Segment Graphs, a question might arise concerning the choise of "12 equally spaced dots." There would need to be 12 dots so that eleven spaces or units could be shown. This would be a good time to consider other scales that also might be suitable.

Children should be aware that the scale that is used affects the "look" of the graph. One can make a set of data look "good" or "bad" depending upon the scale that is chosen. To illustrate this, draw two graphs on the chalkboard. Data similar to the following could be used.

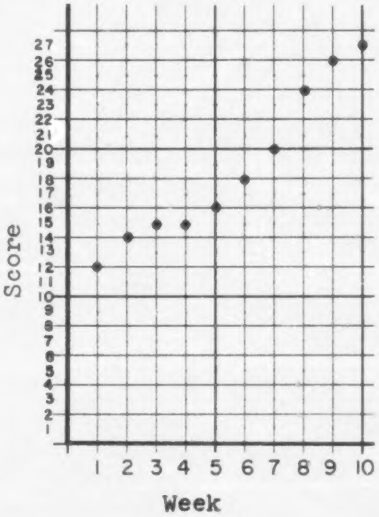
TABLE 1

John's Spelling Scores

Week	1	2	3	4	5	6	7	8	9	10
Score	12	14	15	15	16	18	20	24	26	27



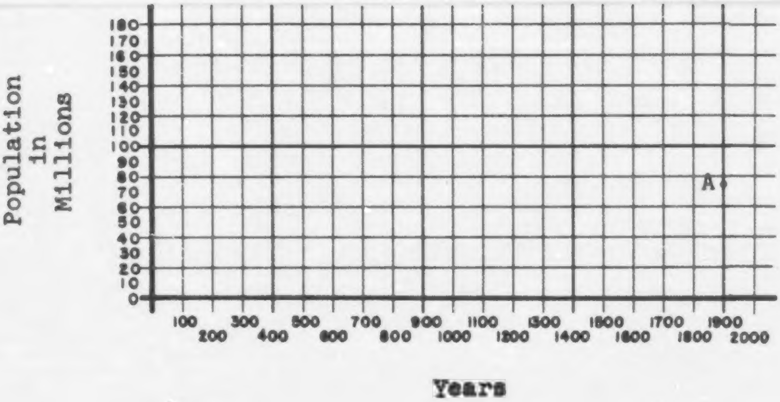
John's Spelling Scores



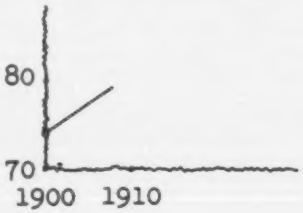
John's Spelling Scores

Both graphs show the same data but Graph 2 makes John appear to be improving more rapidly than does Graph 1. If these dots were connected with line segments, the trend would be easier to see but segments are not appropriate with this data.

Example 1 of the Exploration beginning on page 518 emphasizes that the vertical and horizontal scales need not (and often should not) begin at 0 on the scales. If this were followed in this example, the grid for the graph would appear as shown on the following page, with the first point of the graph at A.



The graph is properly drawn in the Exploration section of the pupils' book. Frequently "torn" edges are used to indicate that only a portion of a graph is being considered.



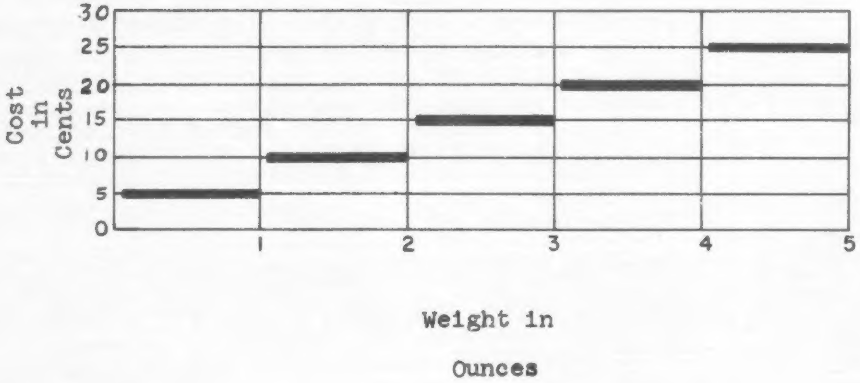
Example 2 of the Exploration emphasizes that segments may be used to connect the points of that example because each point of the segment would represent an estimate of the population at a given time. Example 3 asks the pupil to make some of these estimates.

Exercise Set 3 gives the children opportunities to draw graphs "on their own." It is not expected that this limited practice will develop an exceedingly high degree of skill. The first graph cannot be drawn as a broken-line graph because the troops are separate entities - there is no troop halfway between Troop 1 and Troop 2 or one-fourth of the way between Troop 4 and Troop 5. Thus the graph is just the set of points. In the next section children will learn that a bar graph could be used to show this data.

In exercises 2 and 3 of Exercise Set 4, time should be represented on the horizontal reference line as this is the custom.

Exercise 4 of Set 4 is an example of a "straight line" graph.

The graph of the Braintwister in Set 4 would look like this:



Note that each line segment ends at a whole ounce and begins just to the right of a whole ounce. This is because letters weighing up to and including one ounce require 5¢ postage while letters weighing just "over" one ounce and up to and including two ounces require 10¢ postage, etc.

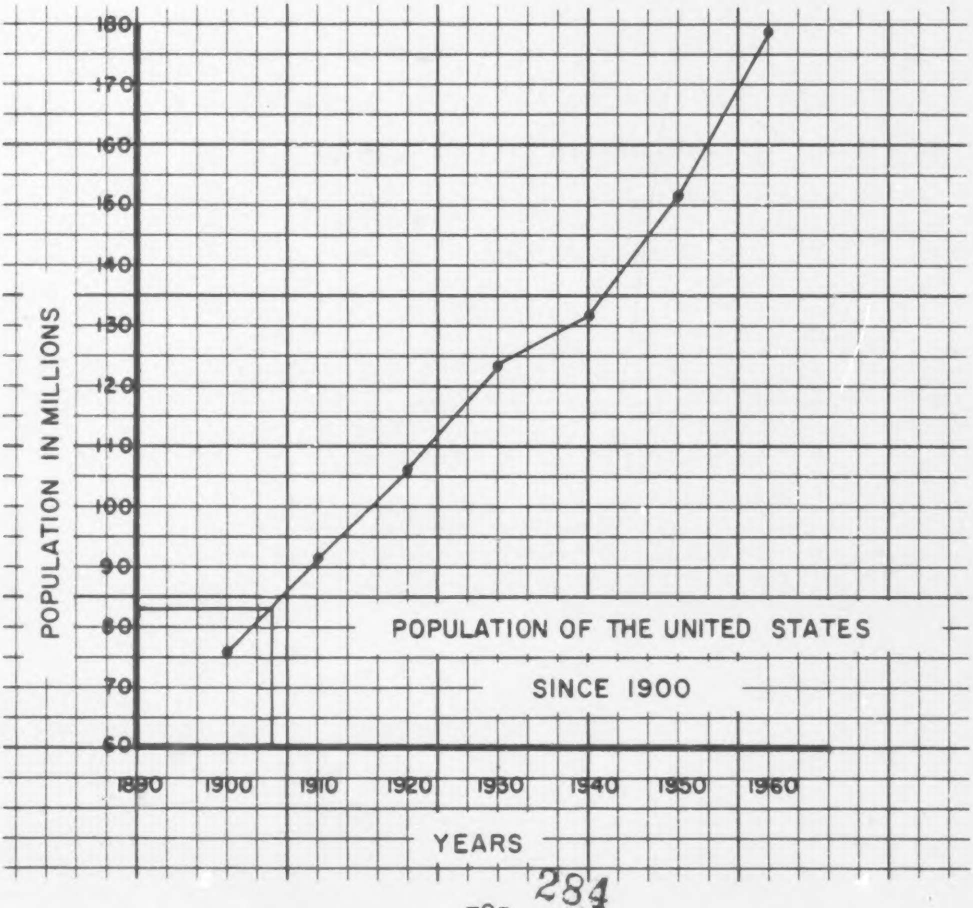
Exploration

The population of the United States for each ten year period since 1900 is shown in the following table. The numbers in the Population column have been rounded to the nearest million.

Census Years	Population in Millions
1900	76
1910	92
1920	106
1930	123
1940	132
1950	151
1960	179

1. The graph of this data is shown following exercise 2 of this set of exercises. The horizontal reference line is usually used as the reference line for time as measured in hours, months, years, or any time interval. Since there are seven 10-year periods represented, we choose 7 equal spaces that use most of the horizontal line segment. The vertical line is used as the reference line for the population. The population difference between 1900 and 1960 is 103 million. (The difference between 179 million and 76 million.) We can let each space on the vertical line represent 10 million in population. We choose 12 equally spaced dots on the vertical line. The bottom dot represents 70 million, the next dot 80 million, etc. The ordered pairs are graphed.

2. It is meaningful in this example to connect the points with line segments. The line segments are drawn. We will call this graph a line segment graph. In this case, the points of the line segments give us estimates of the population between census years. For example, the population in 1905 can be estimated. Find the point on the horizontal axis which represents 1905. There is a vertical line at this point. This vertical line intersects the line segment graph. Follow a horizontal line from this point of intersection to the vertical axis on which the population numbers are shown and estimate the numbers on this axis as carefully as you can.



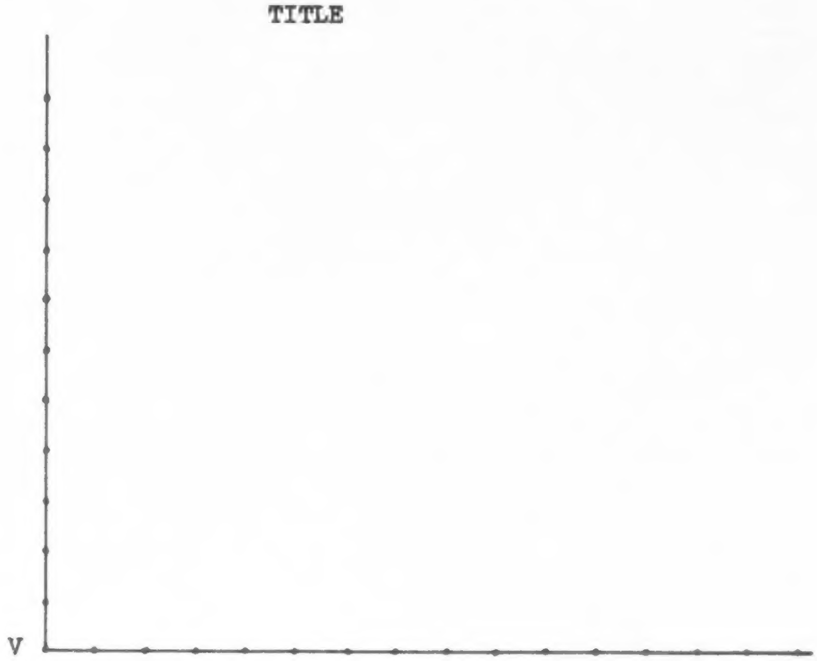
3. From the graph, estimate the population in 1935 and in (127,000,000 and 164,000,000) 1955. Could you estimate the population in 1965? (Not on this graph)

4. Why could we draw a line segment graph for the population data but not for the data showing the vote on the class pet? (There isn't an estimate for a point between a hamster and a garter snake.)

A line segment graph can be drawn only when points of the graph between the endpoints of each line segment represent an estimate of some ordered pair of numbers that could be considered part of the data. Suppose we had taken the population count every year instead of every ten years. Then we could have plotted more points. Our line segment is just a guess at where these points would be located. Thus, you should not use a broken-line graph when you have objects like hamster and turtle represented by points on one of the reference lines. A type of graph that could be used to show the vote on the hamster and the turtle is a bar graph which will be discussed in the next section.

P521

Suppose you are preparing to graph the pairs in the columns (a) and (b) that are shown below. Suppose that you have horizontal and vertical reference lines as shown in the drawing. There is space for the title and numbering of the reference lines. There are 15 spaces on the horizontal reference line and 11 spaces on the vertical reference line.



(a)		(b)	
6,	88	1790	7,400,000
12,	99	1800	9,600,000
18,	110	1810	11,800,000
24,	121	1820	14,000,000
30,	132	1830	16,200,000

1. This part is about column (a). Label the first numbers of the pairs on the horizontal reference line. Begin numbering the vertical scale at 65 and the horizontal scale at 0 starting at the point V. Fill the blanks.

   a) Choose $(\frac{1}{2})$ horizontal spaces to represent 1 and $(\frac{1}{10})$ vertical spaces to represent 1.

   b) The farthest labeled point to the right on the horizontal scale is labeled (30). It is (15) spaces from V.

   c) The highest labeled point on the vertical axis is labeled with (175). It is (11) spaces from V.

   This part is about column (b): Label the first numbers of the pairs on the horizontal reference line. Begin numbering the vertical scale at 72 and the horizontal scale at 1790, starting at the point V.

   d) Choose (3) horizontal spaces to represent 10 and three vertical spaces to represent (1,000,000).

   e) The farthest labeled point to the right on the horizontal scale is labeled (1840). It is (15) spaces from V.

   f) The highest labeled point on the vertical axis is labeled (182). It is (11) spaces from V.

2. On a vertical reference line you have to include numbers between 265 and 530. You have 31 points that you may number. There are 30 spaces.

   What number would you let each space represent? (10)

   How would you number the first point and the last point? (265 and 565)

3. On a horizontal reference line you have to include numbers from 1850 to 4250. You have 12 spaces on the reference line.

   What would you let each space represent? (200)

   What numbers would you use to label the first and last points? (1850 and 4250)

Exercise Set 3

1. The boxes of mint cookies sold by the Girl Scout troops were: Troop 1, 48 boxes; Troop 2, 34 boxes; Troop 3, 72 boxes; Troop 4, 25 boxes; Troop 5, 75 boxes; Troop 6, 51 boxes; Troop 7, 132 boxes; Troop 8, 82 boxes. May a line segment graph be used? (No)

2. Anna weighed 6 pounds at birth, 12 pounds at 6 months, 19 pounds at 12 months, 28 pounds at 18 months, and 34 pounds at 24 months.

   Show her rate of growth using a line segment graph. (See T.C.)

   a) Does the graph you made show Anna's growth to be the same for each six month period? (No)

   b) During which period is her gain in weight the greatest? (Between 12th and 18th month)

3. Max Q. Farmer raised a calf. At the end of a two-year period he had recorded the following information:

   Weight at birth: 70 pounds
   Weight at age 6 months: 400 pounds
   Weight at age 12 months: 600 pounds
   Weight at age 18 months: 1100 pounds
   Weight at age 24 months: 1400 pounds

   Graph these facts using a line segment graph. (See T.C.)

   a) During which period in the calf's life was the greatest gain in weight shown? (Between 12th and 18th month)

   b) How does the gain in weight of the first six months and the last six months compare? (30 pounds more in first 6 months)

   c) How are the two graphs in exercises 2 and 3 similar?

4. Gasoline costs 30¢ a gallon. Complete the table:

Gallons	1	2	3	4	5	6	7	8	9	10
Price	30¢	60¢	? (90¢)	? (120¢)	? (150¢)	? (180¢)	? (210¢)	? (240¢)	? (270¢)	? (300¢)

Draw a line segment graph to show this information.
(For graph, see T.C.)

BRAINTWISTER

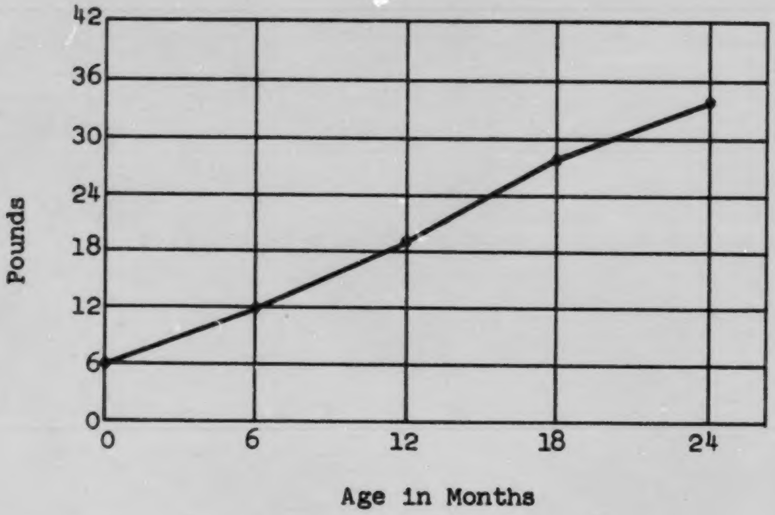
The cost of mailing a first-class letter is 5 cents if the weight of the letter is one ounce or less. The cost is increased by 5 cents for each additional ounce or fractional part of an ounce. For example, it would cost 10 cents to mail a letter weighing $1\frac{1}{3}$ ounces and it would also cost 10 cents to mail a letter weighing 2 ounces. It would cost 15 cents to mail a letter weighing $2\frac{1}{5}$ ounces.

Draw a graph showing the postal rate for first-class letters up to 5 ounces. (For graph, see T.C.)

Read from your graph the cost of mailing a letter weighing $4\frac{1}{2}$ ounces. (20 cents)

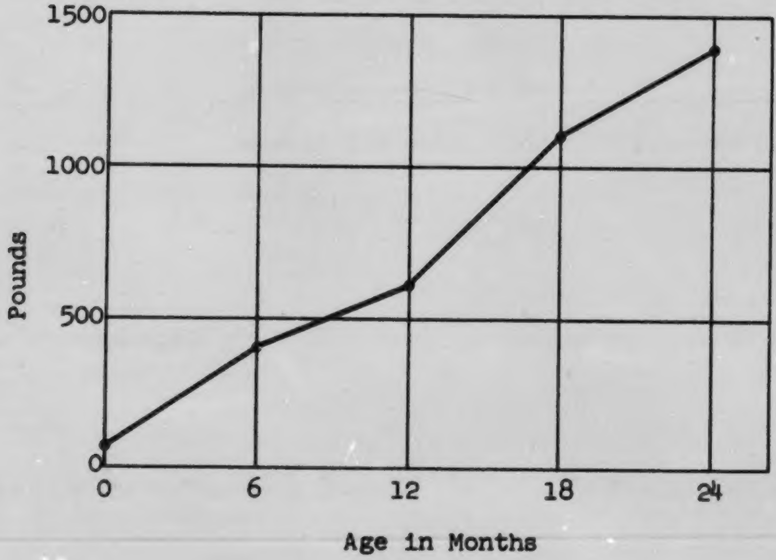
Answer to Set 3, number 2

ANNA'S WEIGHT



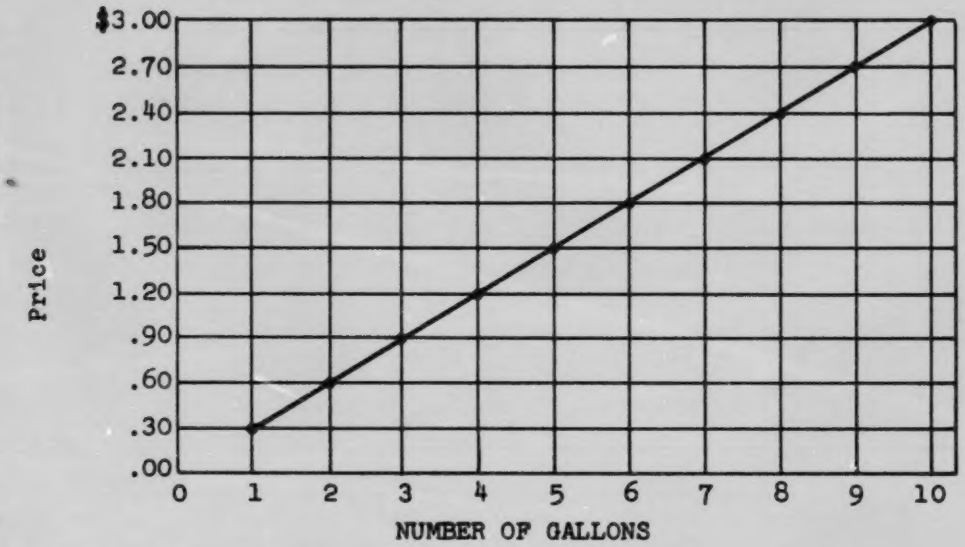
Answer to Exercise 3, number 3

CALF'S WEIGHT



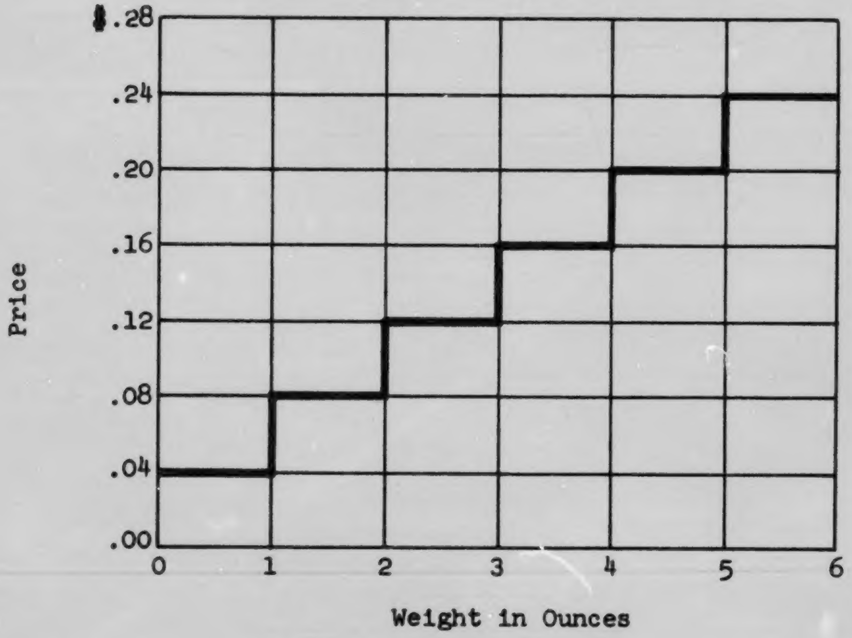
Answer to Exercise Set 3, number 4

COST OF GASOLINE



Answer to Exercise Set 3, Braintwister

MAILING COST



BAR GRAPHS

Objective: To develop the ability to interpret and construct bar graphs and to know when it is appropriate to represent data with this kind of a graph

Vocabulary: Bar graph

Materials: Chalkboard, chalk, straightedge, pencil, paper, graph paper

Suggested Teaching Procedure:

Follow the Exploration as given in the pupil text. Pupils will have their books open.

Many sixth-grade pupils will already have seen and perhaps used this type of graph. Exercise Set 4 gives opportunities to construct and interpret bar graphs.

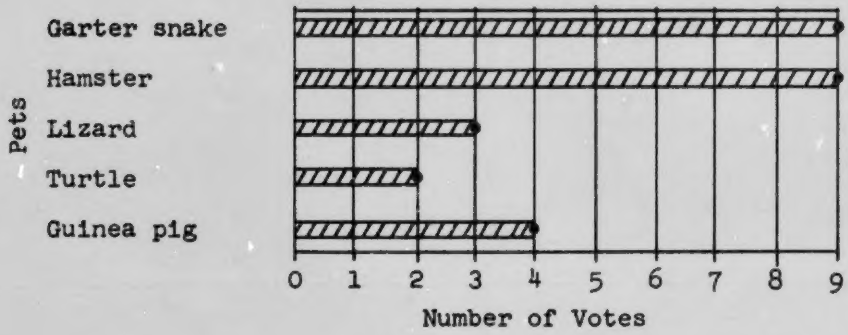
Note that the points are shown on the bar graphs. This is done to emphasize the fact that the points are really the graph of the data and the "bars" just make it easier for us to "see" the graph.

BAR GRAPHS

Exploration

In the last section we found that the results of the election for a class pet could not be shown on a broken-line graph. The entire graph is just the five points, and it is not possible to get any additional points. However, the five dots are difficult to see, so we can draw bars from the reference line for the pets to the points like this:

Bar Graph
of
Vote for Class Pet



This type of graph is called a bar graph. Is a bar graph easier to read than just a graph of an isolated set of points?

A bar graph is used to compare things, such as the vote for each pet.

The bars should be the same width. The spaces between the bars should be the same width, but not necessarily the same as the width of the bars.

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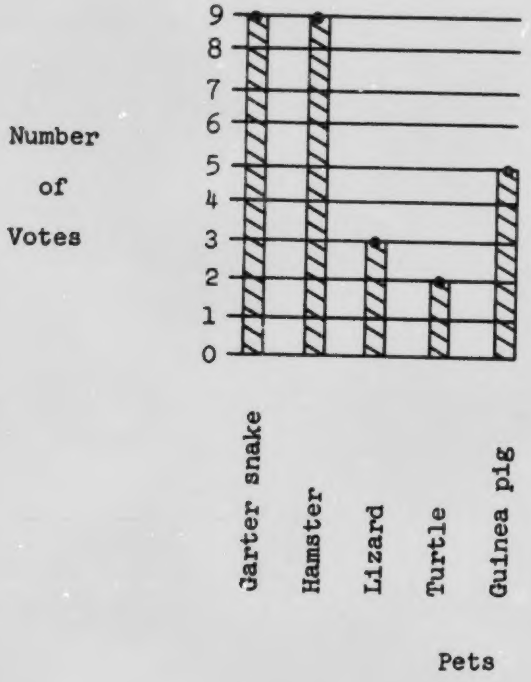
The bars on a bar graph may be vertical rather than horizontal. To make this type of graph for the vote for a class pet, place the names of the pets along the horizontal reference line. Why? The graph might look like this:

Bar Graph
of
Vote for Class Pet



Exercise Set 4

1. A table of newspaper sales is given below. Make a bar graph showing the total papers sold by George, Harry, Jim, and John for one week. (For graph, see T.C.)

TABLE 3

Newspaper Sales for One Week

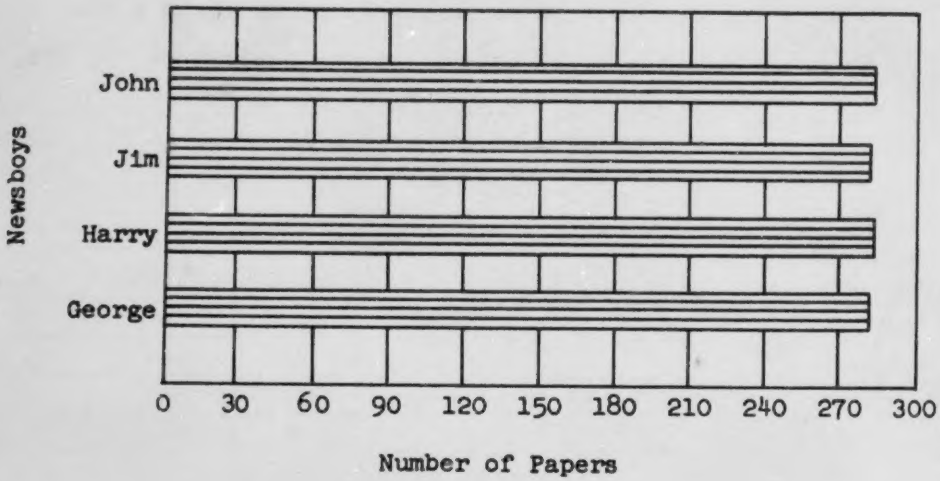
	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday	
George	3	10	18	30	52	70	98	(281)
Harry	28	29	29	30	32	60	75	(283
Jim	15	21	47	47	47	50	54	(281)
John	36	38	38	40	42	44	45	(283)

2. Make a bar graph showing the number of papers sold by George during each day of the week. (For graph, see T.C.)

3. Make a bar graph showing the number of papers sold by each boy on Saturday. (For graph, see T.C.)

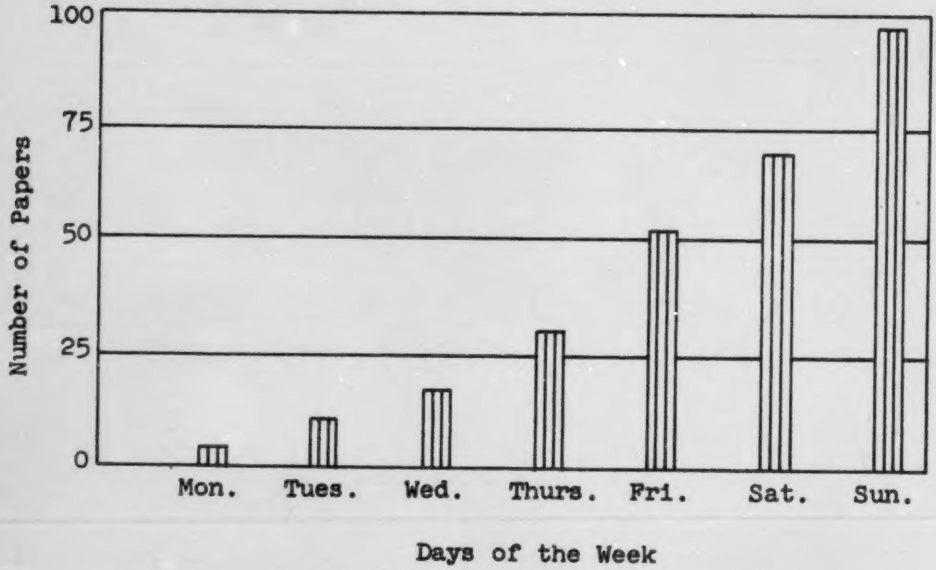
Exercise Set 4, number 1

NEWSPAPER SALES OF FOUR BOYS



Exercise Set 4, number 2

GEORGE'S NEWSPAPER SALES



4. The following American ski-jumping records have been made. Graph this information, using a bar graph. (For graph, see T.C.)

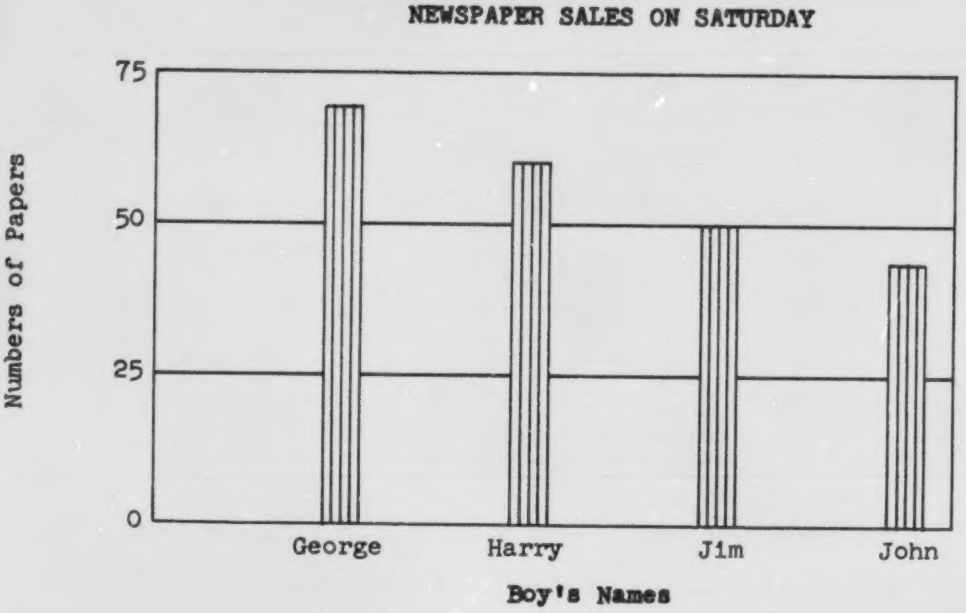
1887	Mikkel Hemmestvedt	37 feet
1905	Julius Kulstad	97 feet
1910	Oscar Gunderson	138 feet
1915	Ragnar Omtvedt	192 feet
1920	Lars Haugen	214 feet
1932	Glenn Armstrong	224 feet
1940	Torger Tokle	228 feet
1950	Art Devlin	307 feet
1951	Austen Samuelstuen	316 feet

The record of 316 feet is the present American record and still stands. Longer jumps have been made in Europe.

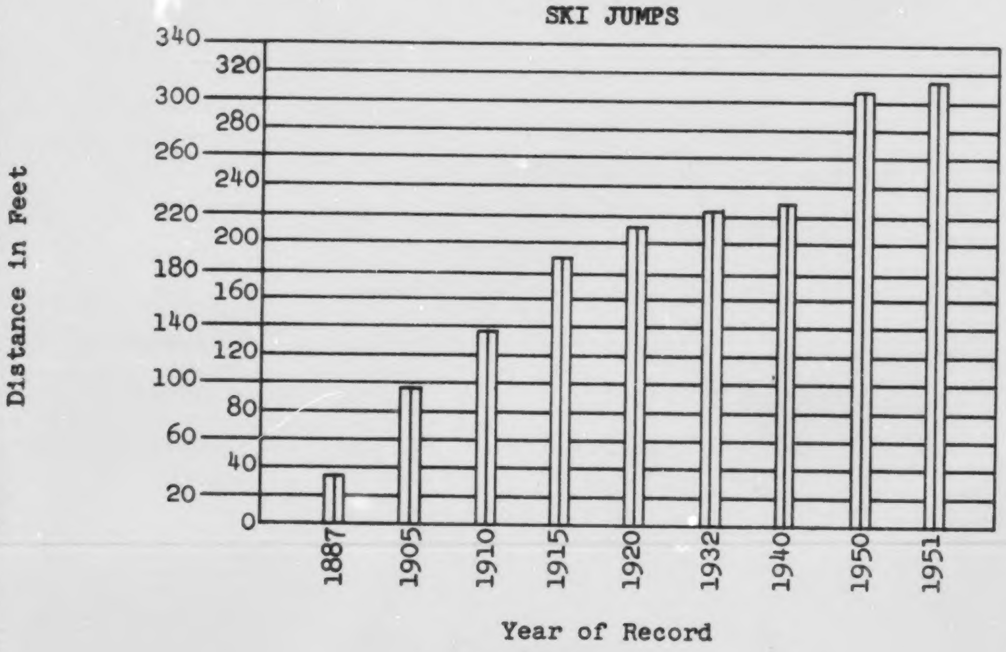
5. Draw a bar graph to show the sale of boxes of mint cookies by these Girl Scout troops. (For graph, see T.C.)

Troop 1 - 48 boxes
Troop 2 - 34 boxes
Troop 3 - 72 boxes
Troop 4 - 25 boxes
Troop 5 - 75 boxes
Troop 6 - 51 boxes
Troop 7 - 132 boxes
Troop 8 - 82 boxes

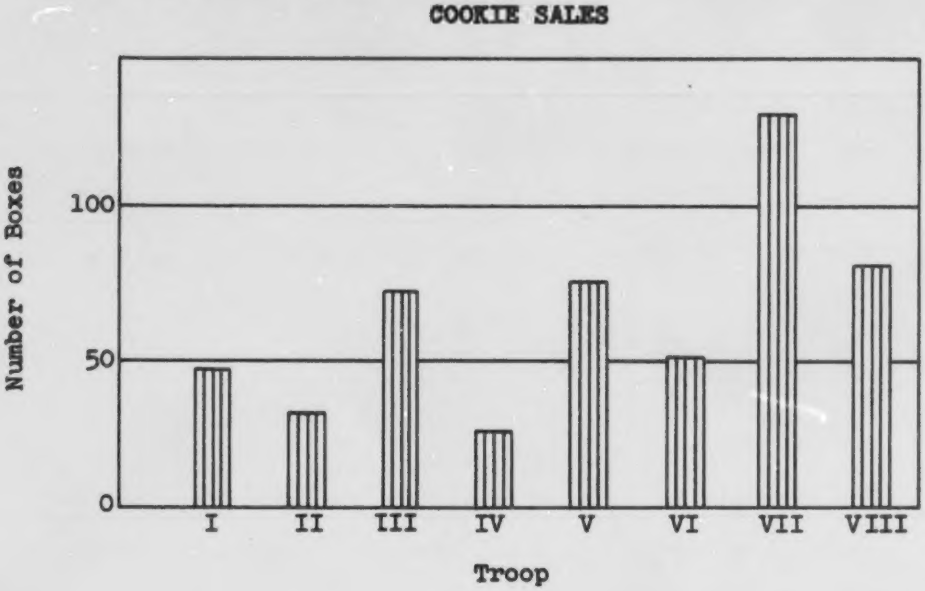
Answer to Exercise Set 4, number 3



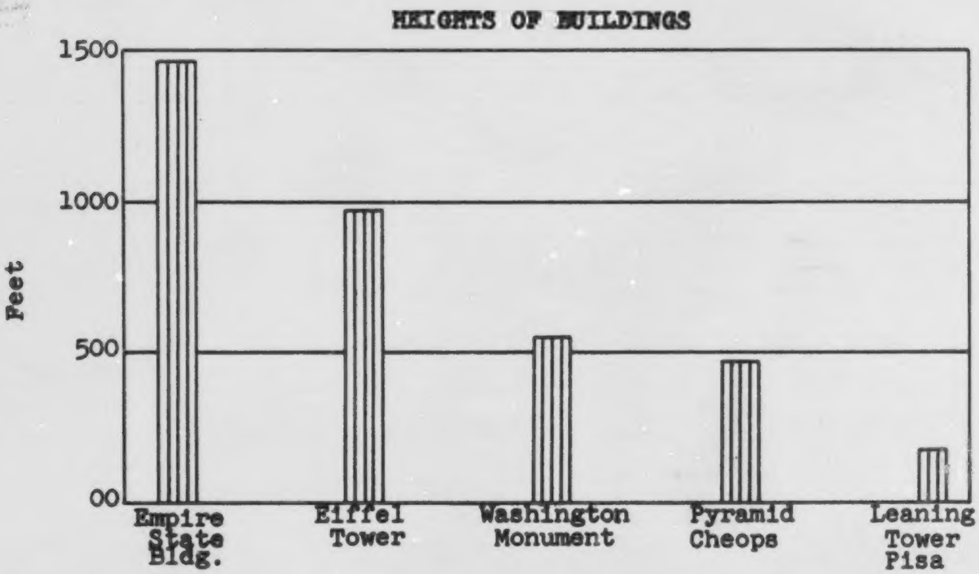
Answer to Exercise Set 4, number 4

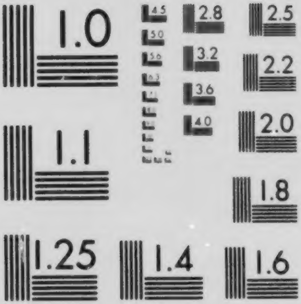


Answer to Exercise Set 4, number 5



Answer to Exercise Set 4, number 6




1.0
4.5
5.0
5.6
6.3
2.8
3.2
3.6
4.0
2.5
2.2
2.0
1.1
1.8
1.25
1.4
1.6

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

6. The Empire State Building in New York City is the world's tallest building. Its height is 1,472 feet. The heights of some other structures are: Eiffel Tower, 984 feet; Washington Monument, 555 feet; Pyramid of Cheops, 480 feet; Leaning Tower of Pisa, 179 feet. Draw a bar graph to illustrate this data. (For graph, see T.C.)

7. The ages of the Presidents at the time they took office were: Washington, 57; J. Adams, 61; Jefferson, 57; Madison, 57; Monroe, 58; J. Q. Adams, 57; Jackson, 61; Van Buren, 54; W. H. Harrison, 68; Tyler, 51; Polk, 49; Taylor, 64; Fillmore, 60; Pierce, 48; Buchanan, 65; Lincoln, 52; Johnson, 56; Grant, 46; Hayes, 54; Garfield, 49; Arthur, 50; Cleveland, 47; B. Harrison, 55; Cleveland, 55; McKinley, 54; T. Roosevelt, 42; Taft, 51; Wilson, 56; Harding, 55; Coolidge, 51; Hoover, 54; F. D. Roosevelt, 51; Truman, 60; Eisenhower, 62; Kennedy, 42.

Make a bar graph to show this data.
(After choosing a suitable scale, the bar graph which could be used may be with either a horizontal or vertical bar. The children's graphs will vary.)

CIRCLE GRAPHS

Objective: To show how a circle graph is used

To develop a way of making simple circle graphs

Materials: Chalk, chalkboard, straightedge, pencil, paper, protractor (for desk and a chalkboard demonstration), compasses (student compass and demonstration compass)

Vocabulary: (Review-words such as circle, diameter, circular region, vertex, right angle, protractor, compass), circle graph.

Suggested Teaching Procedure:

> The first part of the Exploration is essentially a review section but is important as a basis for the ideas used in the Exploration. It is suggested that the teacher and the pupils work through the Exploration step by step keeping the books open. For the greatest benefit from this step by step approach, the pupils should "work out" the answers to the questions and perform the directed activities at their individual desks as the teacher follows the Exploration using the chalkboard. The teacher would save time and avoid delay if the diagram outlines could be made before the class period begins. During the class period, the shaded areas could be filled in and other information could be added.
>
> It is also suggested that the teacher demonstrate a construction wherever the need arises. Usually the pupil text will be in sufficient detail to enable the pupil to make the construction.
>
> To find the measure of an angle when we know the fractional part of a circle, we multiply the rational number by 360. This type of procedure begins at question 8 of the Exploration and is followed through number 11.

In question 12, the purpose of the checking in parts a, b, and c, is simply to re-establish the fact that the sum of the fractional parts of the whole is 1; that the sum of the earnings of all boys is the total sum; and that the sum of the measures of all the angles is 360.

After the Exploration has been thoroughly studied by the pupils, the Exercise Set will provide practice which should be successfully handled by the pupils, with the exception of a few. These exercises should be done on an individual basis by each pupil. After the circle graphs of this Exercise Set are completed, the pupils could compare their work with each other, discuss the common problems in class, and make a display graph for each exercise.

After the completion of this section the pupils would enjoy paging through their textbooks and discussing any circle graphs which they find. The daily papers, financial statements of large companies, and magazines very often have information displayed by circle graphs. Urge the children to be alert for such examples and bring them to class. It adds interest and encourages the child to make use of the new knowledge just acquired.

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CIRCLE GRAPHS

Exploration

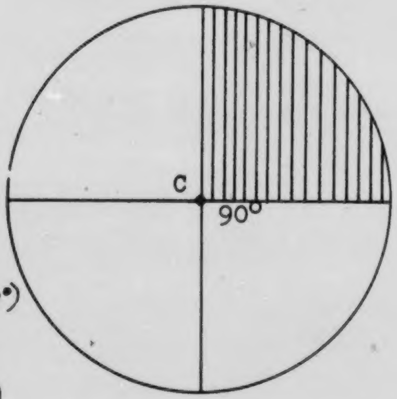
1. What is a circle? (A set of points - a simple closed curve with all its points the same distance from a point called the center.)

2. What is a diameter? (A line segment whose endpoints are points of the circle and whose midpoint is the center of the circle.)

3. Into how many half-circular regions does a diameter divide a circle and its interior? (2)

4. Draw a circle using your compass. Draw a diameter of this circle as in the picture. Label the center of the circle C.

5. Use your protractor to draw another diameter. Draw it so that the angle between the diameters is a 90° angle. How many right angles are formed? (4) What is the measure, in degrees, of a right angle? (90°) Can you find four right angles whose vertices are at C? (yes)



6. Shade one of the regions as in the picture. Is this shaded region one-fourth of the circular region? (yes)

Suppose an angle of one degree is drawn in the shaded region with its vertex at the center of the circle. This angle makes in the shaded region a region shaped like a very small piece of pie. There are 90 of these small pie-shaped regions in the entire shaded region. Why? (The measure of a right angle in degrees is 90.)

7. How many of these small pie-shaped regions would there be in the whole circular region? (360) In half of a circular region? (180) In one-eighth of a circular region? (45)

We can think, then of a circular region separated into 360 regions, each shaped like a small piece of pie.

8. What would be the measure, in degrees, of an angle which forms: a) $\frac{1}{6}$ of a circular region? (60) b) $\frac{3}{10}$ of a circular region? (108) c) $\frac{5}{12}$ of a circular region? (150)

9. How would you find how many of these small pie-shaped regions there would be in any fractional part of a circular region? (Multiply the fraction by 360.)

10. Three boys, Dave, Peter, and Ron, planned to go on a camping trip. They decided to work and earn as much money as possible. Then they would put all the money together. Dave earned \$4, Peter earned \$6, and Ron earned \$8. This made a total of \$18.

a) What part of the \$18 did Dave earn? $\left(\frac{2}{9}\right)$

b) What part of the \$18 did Peter earn? $\left(\frac{1}{3}\right)$

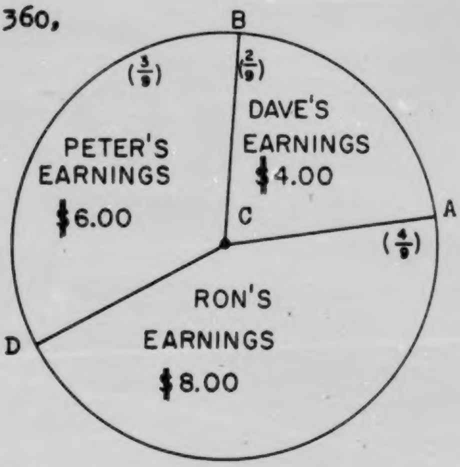
c) What part of the \$18 did Ron earn? $\left(\frac{4}{9}\right)$

Dave earned $\frac{4}{18}$ or $\frac{2}{9}$ of the total money, Peter earned $\frac{6}{18}$ or $\frac{1}{3}$ of the money, and Ron earned $\frac{8}{18}$ or $\frac{4}{9}$ of the money.

11. Suppose the boys wished to show by a circle graph what part of the $18 each gave. How could they do it? They would probably follow these steps:

a) With the compass, draw a circle of a convenient size. We will show each boy's share of the earnings by a "piece of pie." The entire "pie" represents the total earnings.

b) To show Dave's earnings we can multiply $\frac{2}{9}$ by 360, since there are 360 pie-shaped regions in a circular region. ($\frac{2}{9} \times 360 = \frac{720}{9} = 80$)



c) With your protractor draw an angle of 80° with vertex at the center of the circle. You have divided the whole pie into two regions. Which region represents Dave's earnings?

(To show Peter's earnings we multiply $\frac{1}{3}$ by 360. $\frac{1}{3} \times 360 = 120$. With a protractor measure an angle of 120°. The angle should have a ray common with the previous angle.)

12. Ron's earnings should be represented by the remaining piece of pie. Angle ACD is an angle at the center of the circle equal to $\frac{4}{9}$ of 360° or 160°. Is it? (yes)

Let us do a little checking:

a) How much is the sum: $\frac{2}{9} + \frac{3}{9} + \frac{4}{9}$? $\left(\frac{9}{9} = 1\right)$

b) How much is the sum of \$4.00 and \$6.00 and \$8.00 ? (\$18.00)

c) How much is the sum of 80° and 120° and 160° ? (360°)

The sum of the fractional parts of the circle should equal 1.

The amount of money given by the boys should equal \$18.00.

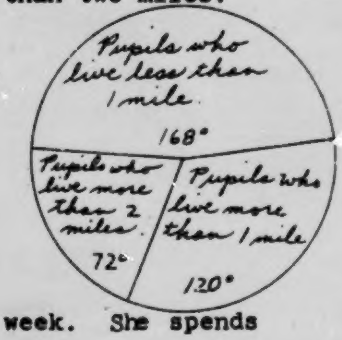
The sum of the measures of all the angles should equal 360.

13. Does our circle graph show that these three statements a), b), and c) in exercise 12 are correct? (yes)

Exercise Set 5

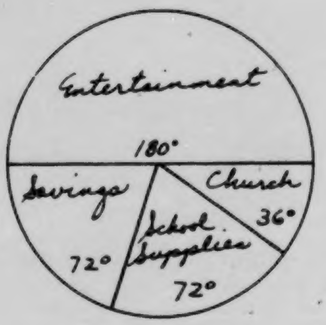
1. There are 30 pupils in the sixth grade at Lincoln School. There are 14 of these pupils that live less than one mile from school, 10 that live more than one mile and less than two miles, and 6 that live more than two miles.

   Represent these data on a circle graph.



2. Mary gets an allowance of one dollar a week. She spends 50¢ each week for entertainment (such as movies). She saves 20¢ in her savings bank each week and gives 10¢ to the church. She spends 20¢ each week for school supplies.

   Draw a circle graph which shows the percent of her allowance that Mary Uses for each purpose.



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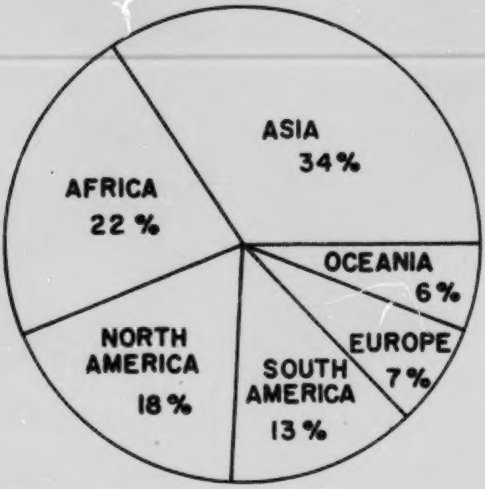
3. Below is a circle graph. Study it and then answer the questions which follow it.



AREA OF CONTINENTS AS PER CENT OF WORLD LAND AREA

a) Europe and Asia combined make up what per cent of the world land area? (41%)

b) Is the combined area of North and South America as great as that of Asia? (No)

c) Is the land area of North America more or less than $\frac{1}{5}$ of the world land area? (less than $\frac{1}{5}$ of the world land area)

d) Should the measure of the angle which determines the region representing Asia's area be more or less than 120? (More than 120)

PICTOGRAPHS

Objective: To show how to interpret and make a pictograph

To show some uses for pictographs

To develop an awareness that information may be inaccurately portrayed by means of a pictograph

Materials: Chalk, chalkboard, straightedge, pencil, paper

Vocabulary: Pictograph

Suggested Teaching Procedure:

The pupil text is quite sufficient in this section. However, some ideas may have to be "pushed" in order to receive the emphasis necessary to point out these ideas:

1. Pictographs use appropriate symbols or pictures which may represent a definite quantity; that is, each symbol is the same shape and size.
2. The symbols may be the same shape but vary in size to give the desired effect.
3. When only a part of a symbol is used, such as 1/3 of a telephone to represent 1,000,000 phones, the children should realize that estimation plays a great part in the interpretation of the graph.
4. The pictograph may be used to give a picture not entirely true as is the case in the graph showing the heights of Bill and John.
5. The skill of rounding off numbers is used in order to make for ease in choosing a scale.
6. Some pictographs use a variety of symbols; other pictographs use the pictures themselves.

In the Braintwister at the end of this section, only the western states named were chosen so that the exercise would not be too difficult. If the area of the state is drawn in proportion to the total population of that state California would have the largest area of the western states. Texas would not be the largest; Nevada and Wyoming would be very small.

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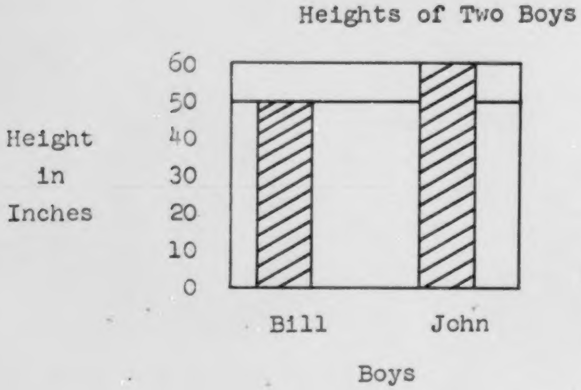
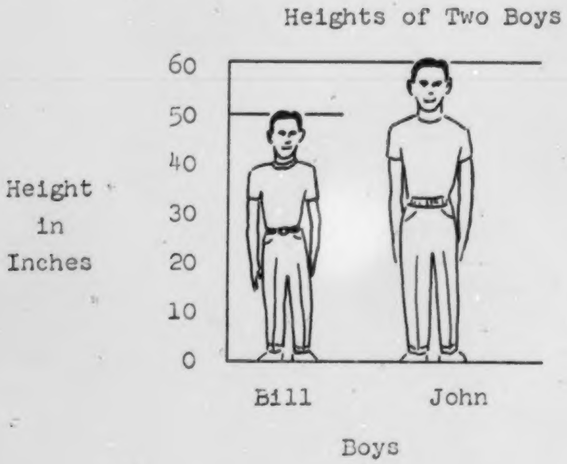
PICTOGRAPHS

Exploration

John was 60 inches tall and Bill was 50 inches tall. John decided to draw a bar graph to impress Bill with how much taller he was. The graph looked like this:



John decided this graph wouldn't impress Bill very much so he decided to draw pictures of himself and Bill to replace the bars. Now the graph looked like this:



1. Does the graph make the difference in heights seem greater than it actually is? (yes) Why? (All dimensions increase, not just height.)

When a graph shows pictures of the objects represented by the data, as in the last example, it is called a pictograph. Some pictographs may give an unfair picture of the data being presented. For example, the pictograph of the heights of Bill and John shows John is not only taller, but is also larger! That is, his shoulders are wider, his head is larger, and so on.

2. In 1951, the number of telephones in use in the United States was 66,645,000 and in Europe the number was 37,593,000. Round these figures off to the nearest million and draw a bar graph. Then draw a pictograph by drawing two telephones, each one as tall as the length of the corresponding bar of the bar graph. Does this pictograph represent a fair picture of the data? (No, because more than just the heights of the telephones are different.)

3. Another way of drawing a pictograph of the number of telephones in various parts of the world would be to draw one telephone for each 3,000,000 telephones. Africa, with about 2,000,000 telephones would have to be represented by a picture of $\frac{2}{3}$ of a telephone. This pictograph is shown on the following sheet. Do you get a better understanding of the telephones in the United States and Europe from the pictograph of exercise 2 or exercise 3? (Exercise 3)

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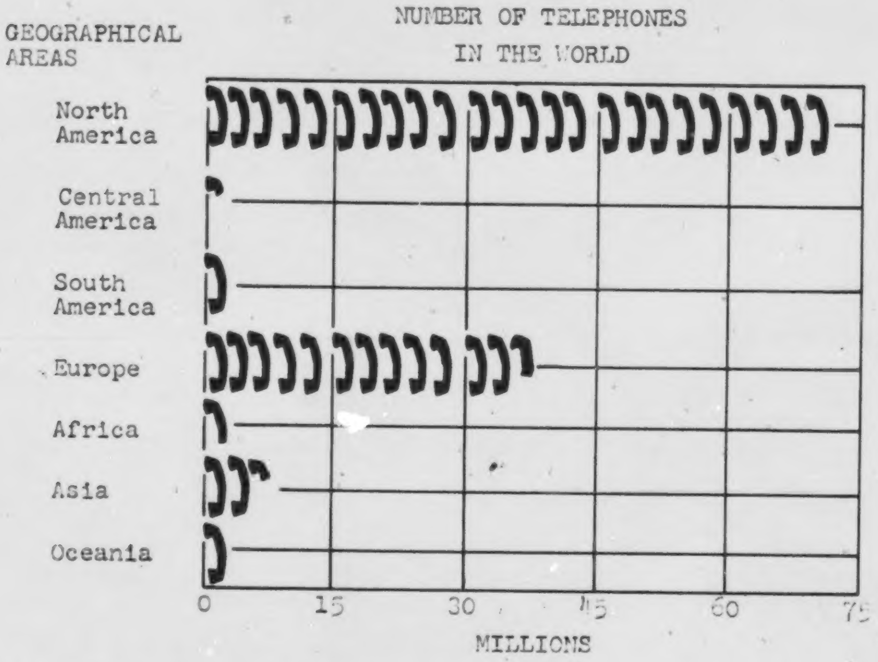
World Telephone Statistics

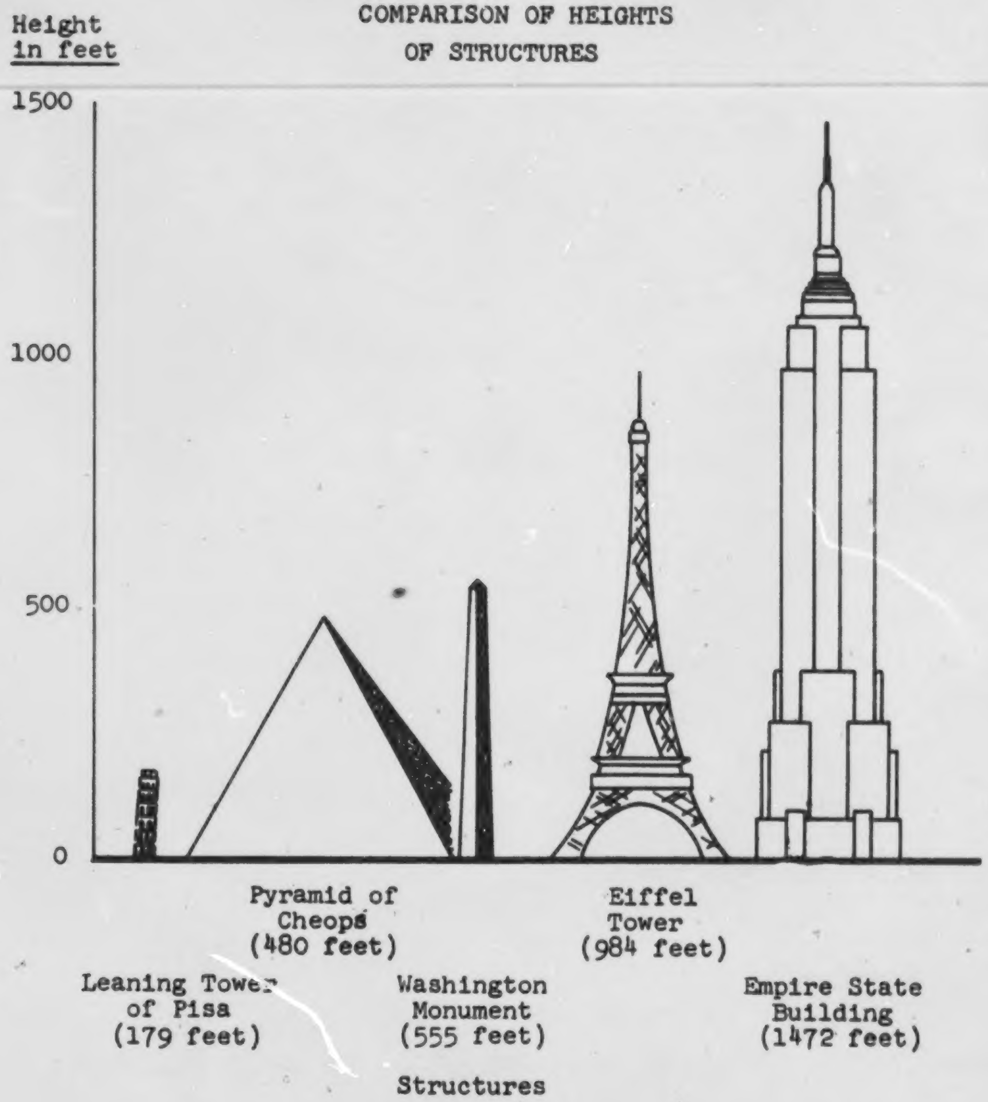
Area	Total Number of Telephones
North America	71,803,700
Central America	910,800
South America	2,999,600
Europe	37,593,900
Africa	1,768,600
Asia	6,855,500
Oceania	2,867,900

Round these numbers to the nearest one million. If we draw one picture for each 3 million telephones, how many pictures of telephones will we draw for each country?



(Each symbol represents 3,000,000 telephones.)

5. A pictograph for Problem 6 of Exercise Set 5 on the height of the Empire State Building and other structures is shown below.



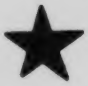
COMPARISON OF HEIGHTS OF STRUCTURES

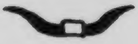
Do you like the pictograph better than the bar graph?

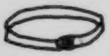
(Answers will vary.)

Exercise Set 6

1. On Major Cassidy's Dude Ranch there are 22 horses, 18 milk cows, 4 dogs, and 12 cowboys. Let us choose a symbol to represent horses, another to represent cows, and still other symbols for dogs and cowboys. The following symbols will be used:

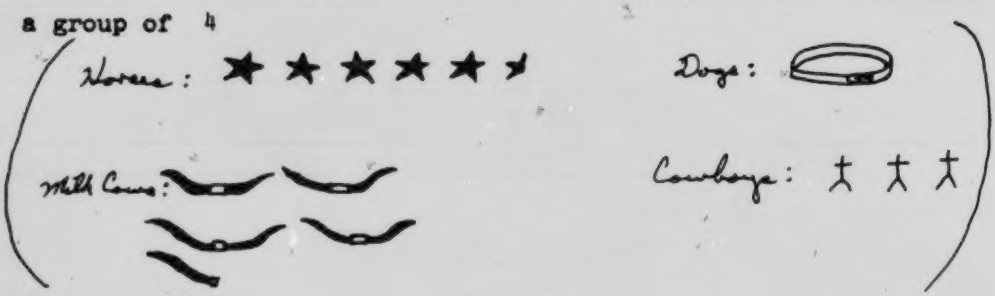
A star represents horses 

A set of horns represents cows 

A dog collar represents dogs 

A stick-man represents cowboys

Make a pictograph of this data. Let each symbol represent a group of 4



2. Draw a pictograph to compare the population of the United States in 1900, 1930, and 1960. The 1900 population was 76 million, the 1930 population was 123 million, and the 1960 population was 179 million. (graphs will vary)

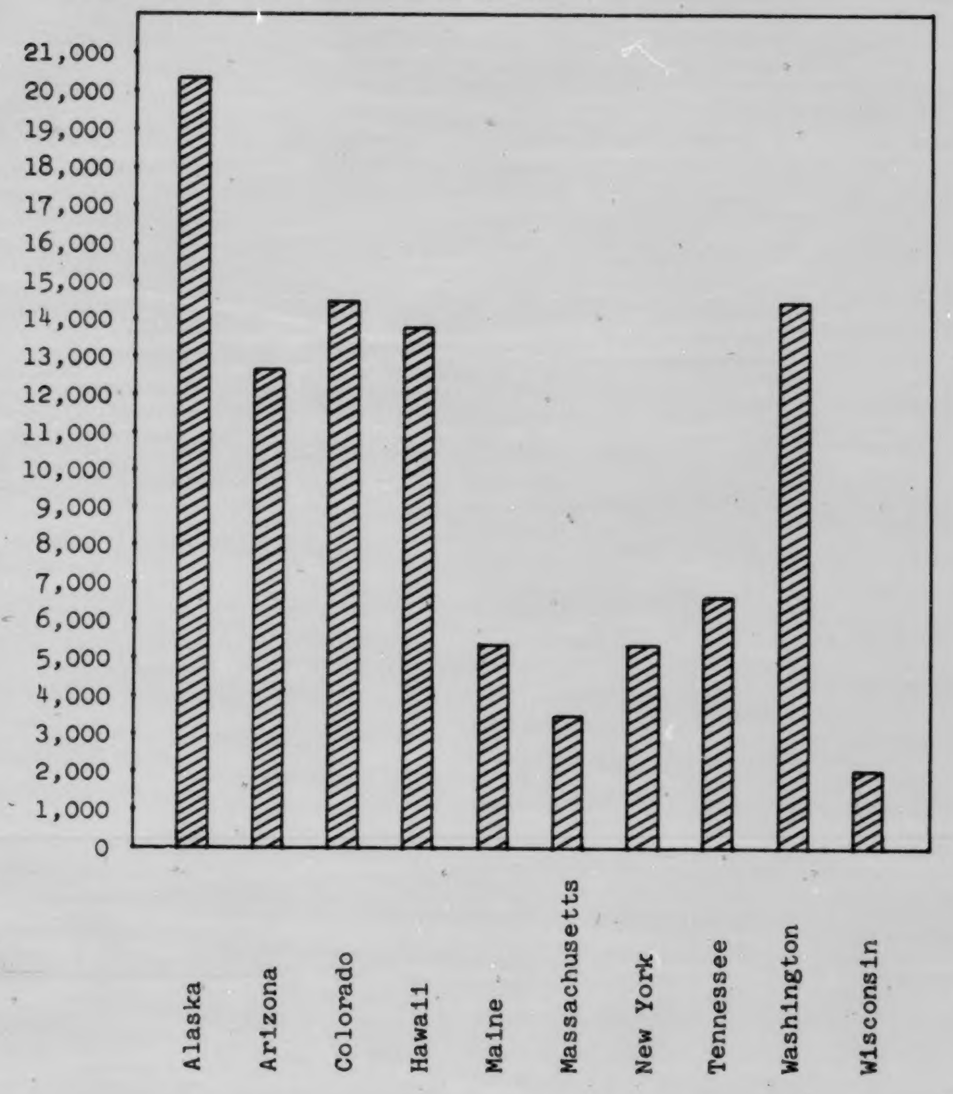
3. The highest altitude in each of these states is given. Round off each of the numbers to the nearest multiple of 100, then draw a bar graph. Put the names of the states along the horizontal reference line. (For graph, see T.C.)

Alaska	20,320 feet	(20,300)
Arizona	12,670 feet	(12,700)
Colorado	14,431 feet	(14,400)
Hawaii	13,796 feet	(13,800)
Maine	5,268 feet	(5,300)
Massachusetts	3,491 feet	(3,500)
New York	5,344 feet	(5,300)
Tennessee	6,642 feet	(6,600)
Washington	14,410 feet	(14,400)
Wisconsin	1,941 feet	(2,000)

BRAINTWISTER

Using the 1960 census figures, draw a rough map of western United States. Let the size of each state depend upon its population. That is, the states which have the largest population should be drawn larger than states which have smaller populations. Use Washington, California, Oregon, Montana, Idaho, Nevada, Wyoming, Arizona, New Mexico, Texas, and Colorado.

Answer to Exercise Set 6, number 3



MEASURES OF CENTRAL TENDENCY

Objective: To develop the idea that description of data may be briefly done by use of an average

To teach the pupils that there are three averages: arithmetic mean, median, and mode

To show that in some cases one type of average gives a better description than another

Materials: Chalkboard, chalk, pencil, paper

Vocabulary: Average, arithmetic mean, median, mode

Suggested Teaching Procedure:

Point out to the pupils that there are three types of averages. The first type known as "arithmetic mean" is simply found by dividing the total sum by the number of entries. Children have been finding this average in various cases before. It should be remembered that this type of average is not always the fairest. The average chosen to dedescribe the data must describe it most accurately.

The advantages and disadvantages of using the median and mode should be emphasized. Questions 1-c through 1-j of the Exploration are written with this in mind as well as to develop the concepts of median and mode.

In the second part of the Exploration the purpose is two-fold:

1. To give practice in computing the arithmetic mean and median, and in determining the mode.
2. To compare these measures with respect to one another.

The pupil text carries out these two purposes very well. Wherever likenesses or differences occur among the three averages they should be pointed out.

The pupils will probably need help in understanding that more than one mode may occur or if several items occur only once each, there isn't a mode established.

In computing the median when an even number of entries occur the teacher may have to reexplain using a different set of numbers. Most of the pupils, however, will be able to comprehend from the explanation contained in the pupil's text.

The exercises are designed to point up the values and shortcomings of the three types of averages. The answers to these exercises will contain helpful hints and suggestions along this line.

MEASURES OF CENTRAL TENDENCY

Exploration

Once the data have been organized, the next problem is to try to find a number which will describe the data and help us understand it. A number which tells us something about how near the number pairs are to some central number pair is called a measure of central tendency, or an average. There are several different types of averages.

1. Here are the scores of Bill, John, and Mike, on five tests:

	Test 1	Test 2	Test 3	Test 4	Test 5
John	80	75	80	100	80
Bill	40	80	60	80	80
Mike	25	80	30	81	82

a) For each boy, arrange the scores from smallest to largest.

Joh	75	80	80	80	100
Bill	40	60	80	80	80
Mike	25	30	80	81	82

b) How would you find the "average" of John's scores?

Probably you found the "average" by adding the scores together and dividing by the number of scores, like this:

$$\frac{75 + 80 + 80 + 80 + 100}{5} = 83$$

Are many of the scores in the table "near" 83? The scores of 75, 80, 81, 82 are not far from 83. This type of average is called the arithmetic mean. The word "arithmetic" in "arithmetic mean" is not pronounced the way you usually pronounce the word. Look in the dictionary for the correct pronunciation. The arithmetic mean is an average computed by using arithmetic (addition and division are used). Thus you should be able to remember the name of this average.

c) Find the arithmetic mean of Bill's scores and of Mike's scores. (Bill 68, Mike $59\frac{3}{5}$)

d) There may be some other "average" that would make Bill's grades look better.

The number that occurs most frequently in a set of numbers is another type of average called the mode. We think of the most popular type of dress or hat as this year's mode or style; hence, the choice of the word mode. If no item occurs more than once, there is no mode.

e) What is the mode of Bill's scores? (80) Is it better than the arithmetic mean of his scores? (yes)

f) What is the mode of John's scores? (80) Is it better than the arithmetic mean of his scores? (No, the arithmetic mean of John's scores is 83.)

g) There is still a third type of average that might be useful in comparing the scores.

When the numbers of a set are arranged in order of increasing or decreasing size, the median is the number that is in the middle. There are just as many numbers below the median as above. The word "median" means "middle." By associating these two words you should be able to remember that the median score is the middle score.

h) What is the median of Mike's scores? (80) Is it better than the arithmetic mean of his scores? (yes)

i) Make a table and list the arithmetic mean, median, and mode for the scores of John, Bill, and Mike.

	Mean	Median	Mode
John	83	80	80
Bill	68	80	80
Mike	$59\frac{3}{4}$	80	none

j) Which type of average seems to be the fairest for describing the various sets of test scores? (arithmetic mean)

2. In example 4 of the Exploration on Organizing Data near the first page of this chapter you were asked to make a table showing the number of boxes of cookies sold by eight Girl Scout troops. Your table probably looked like this:

TABLE 4

Cookie Sales
by
Girl Scout Troops

Troop	Mint	Chocolate	Vanilla
1	48	63	35
2	34	27	30
3	72	51	40
4	25	14	12
5	75	39	51
6	51	62	37
7	132	98	99
8	82	103	76

The arithmetic mean of the numbers of boxes of mint cookies sold by the different troops is

$$\frac{48 + 34 + 72 + 25 + 75 + 51 + 132 + 82}{8} = 64\frac{7}{8}$$

a) Which three troop's sales of mint cookies were nearest the arithmetic mean? (Troop 3, Troop 5, Troop 6)

b) Arrange the numbers of boxes of mint cookies sold from the smallest number to the largest number.
(25, 34, 48, 51, 72, 75, 82, 132)

To find the median number sold, we have to find the median of eight numbers. Because 8 is an even number, there isn't any one middle number. So consider the two middle numbers. These are 51 and 72. Any number between 51 and 72 would be a number such that four sales were less (25, 34, 48, 51) and four greater (72, 75, 82, 132). Usually, in cases like this, the arithmetic mean of the two middle numbers is chosen as the median. The arithmetic mean of 51 and 72 is

$$\frac{51 + 72}{2}, \text{ or } \frac{123}{2}, \text{ or } 61\frac{1}{2} .$$

The median is $61\frac{1}{2}$.

c) Which troop's sales of mint cookies were nearest the median? (Troop 6 and Troop 3)

d) Find the arithmetic mean of the sales of chocolate cookies and of vanilla cookies. (chocolate $57\frac{1}{8}$, vanilla $47\frac{1}{2}$)

e) Which two troop's sales of chocolate cookies were nearest the arithmetic mean? (Troop 6 and Troop 3)

f) Find the median of the sales of chocolate cookies and vanilla cookies. (chocolate $56\frac{1}{2}$, vanilla $38\frac{1}{2}$)

g) Which two troop's sales were nearest the median?
(Chocolate: Troops 6 and 3
Vanilla: Troops 6 and 3)

P547

Exercise Set 7

1. Farmer Jones can grow a good garden if about four inches of rain falls each month during the growing season from May until October. The rainfall during these months last year was:

May	1 inch
June	0 inches
July	0 inches
August	1 inch
September	10 inches
October	12 inches

Compute the arithmetic mean (4) and the median. (1) Is the average monthly rainfall 4 inches, using one of these averages? (yes) Do you think Farmer Jones had a good garden? (no) Why? (There are too many months when the rainfall was not close to the average.)

2. The principal announced that the average number of students in the fifth and sixth grades was 25. By average he meant the arithmetic mean. There was one class of each grade. There were 15 pupils in the fifth grade.

How many pupils were in the sixth grade? (35)

3. The temperature in degrees at noon on the first day of each month in the town of York were recorded as follows:

January	12	July	105
February	12	August	105
March	32	September	62
April	55	October	45
May	76	November	17
June	105	December	12

Find the arithmetic mean $(53\frac{1}{6})$ and the median. (50) Do these averages indicate that York would have a pleasant year-round temperature? (yes) Find the mode. This distribution has two modes so list both of them. (12, 105) Now do you think York has a pleasant year-round temperature? (No)

4. Table 3 near the front of this chapter shows the newspaper sales of four newsboys.

a) Find the arithmetic mean of the weekly sales for George, Harry, Jim, and John. (George $40\frac{1}{7}$, Harry $40\frac{3}{7}$, Jim $40\frac{1}{7}$, John $40\frac{3}{7}$)

b) Find the median of the weekly sales for each of the newsboys. (respectively: 30, 30, 47, 40)

c) Which average shows better how busy the newsboys usually are? (median)

5. From the data on the ages of the presidents at the time they took office (Exercise Set 5, exercise 7), find the arithmetic mean and median. (mean is $54\frac{19}{35}$, median is 55.)

Chapter 9

SETS AND CIRCLES

PURPOSE OF UNIT

Prior to the study of this unit the pupils will have studied the unit entitled Concept of Sets. This unit reviews briefly some of the ideas of that unit and then deals particularly with the intersection and union of sets. One principal objective in the study of sets is to establish recognition of the intersection and union of two or more given sets. Emphasis will be placed on the intersections and unions of just two sets but this should not obscure the possibility of more than two being considered. The use of Venn Diagrams is encouraged as an aid in selecting intersections and unions of given sets. The applications in the exercises are expected to stress and clarify the geometric concepts which have been studied previously as well as give the pupils some practice in using the language of sets and in making Venn Diagrams.

The main objectives in the part of this unit which is devoted to circles is the renewed emphasis on a circle as a set of points. The points which constitute a circle are all equally distant from one point which is called the center of the circle. Recognition of the meaning of arc, radius, diameter, interior region and exterior region are important and the pupils should learn to recognize these in a representation of a circle and use the words with facility. The use of the compass in drawing the representation of a circle should be reviewed or taught anew if the pupils have not used it previously. The pupils should be encouraged to look for representations of circles in familiar things around them, e.g., church windows, wheels, edges of tin cans, etc. The inscribed angles are introduced to make clear the meaning of inscribed angle and to prepare the way for their use later. The principal application of the inscribed angle is that, "An angle inscribed in a semicircle is a right angle." This conclusion is expected to be reached intuitively and no mention is meant to be made here of the measure of an arc in terms of degrees.

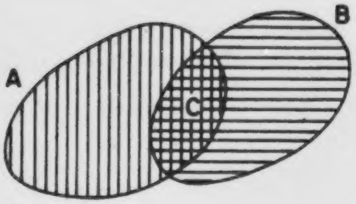
MATHEMATICAL BACKGROUND

A portion of the mathematical background needed for *Sets and Circles* is found in the Teachers' Commentary for the *Concept of Sets* in a previous chapter. A few brief remarks here on the intersection and union of sets and a short discussion of circles will be the main effort in this section.

Intersection of Sets. The intersection of two or more sets is a set although it may have no members. If the intersection set has no members, then the intersection is called the empty set, or the null set. The symbol for the empty set which is used frequently is the Greek letter $\emptyset$. But we shall use the symbol $\{\ \}$ for the empty set since it seems to suggest a set with no members. By definition, the intersection of two sets is a set whose members are in both the given sets. Hence, the intersection of any set, with the empty set is the empty set itself. If A is any set, we can write

$$A \cap \{\ \} = \{\ \}$$

Diagrams, called Venn Diagrams, are very useful in giving a pictorial representation of intersections. Each of the following diagrams represents the intersection set indicated below the diagram. The regions are not meant to be circular regions although circular regions are very often used in Venn Diagrams. Indeed, it is better if the regions are not circular since pupils might make some unwarranted inferences about relations between sets and circles when none are anticipated. In each diagram the portion representing the intersection set is shaded. If no portion is shaded, then the intersection is the empty set.



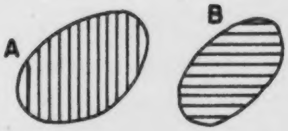
Example:

A = set of boys in your school who have blue eyes.

B = set of boys in your school who are 12 years old.

C = set of boys who are 12 years old and who have blue eyes.

$A \cap B = C$



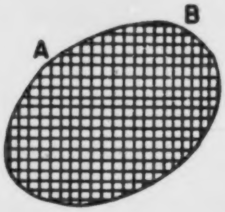
Example:

A = set of boys in Grade 5.

B = set of boys in Grade 8.

The intersection is the empty set since no boys are in both Grades 5 and 8.

$A \cap B = \{ \}$



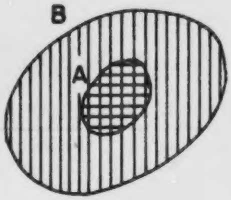
Example:

A = {a, b, c, d, e}

B = set of the first five letters of the alphabet.

Set A = Set B

$A \cap B = A$ or $A \cap B = B$



Example:

B = set of people who live in U.S.A.

A = set of people who live in Ohio.

All of the people who live in Ohio live in U.S.A.

$A \cap B = A$

If all members of a set A are also members of a set B, then A is called a subset of B. (We may write this $A \subseteq B$, or $B \supseteq A$. This notation is not used in the pupil's material.) By this statement either of two equal sets is a subset of the other. For example, if $A = \{a, b, c, d, e\}$ and B = the set of the first five letters of the alphabet, then A is a subset of B and B is a subset of A. But if all members of A are in B and B contains other members which are not in A, then A is a proper subset of B. For example, if A is the set of people who live in Ohio and B is the set of people who live in U. S. A., then A is a proper subset of B. We may write $A \subset B$.

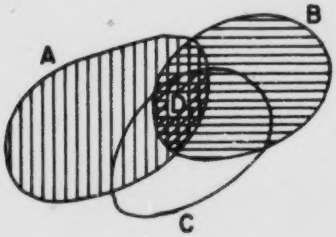
Venn Diagrams are useful in illustrating the relationships between more than two sets although, of course, the diagrams may be a bit more complicated. Consider the diagram for the following three sets:

Let the oval region A represent the set of boys in your school.

Let the oval region B represent the set of boys named John.

Let the oval region C represent the set of boys who have blue eyes.

Then the region D represents the set of boys in your school who are named John and who have blue eyes.

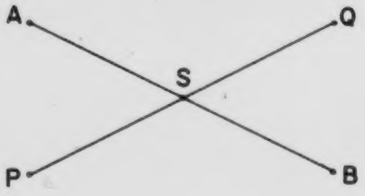


$$(A \cap B) \cap C = D$$

The diagram shows that (1) some of the boys in your school are named John, (2) some of the boys in your school have blue eyes, and (3) some of the boys in your school who have blue eyes are named John. The shaded region D represents the possibility described in (3).

In the next few examples of intersections of sets, the drawings are the representations of geometric figures; they are not part of Venn diagrams. Care will be needed with the pupils to help them distinguish between curves used in Venn diagrams and curves and other geometric figures which represent the sets whose intersections (or unions) are under consideration. The closed curves in a Venn diagram are devices used to assist in thinking of the sets of elements that are involved in the discussion. The curves in a Venn diagram and their interiors are not the sets under consideration. Such a curve is drawn to help in picturing the members as belonging to a set.

In a drawing of geometric figures in which their intersections (or unions) is asked for, the geometric figures are themselves the sets, i.e., the points of the geometric figures are the elements of the sets. These sets of points may be the points of a triangle, the points in the interior of a triangle, the points of a circle, the points of the interior of a circle, the points of a line segment, etc. In a Venn diagram the closed curves are an *aid to thinking* of the sets; in a drawing involving the intersections (or unions) of geometric figures, the points of the figures *are* the sets.



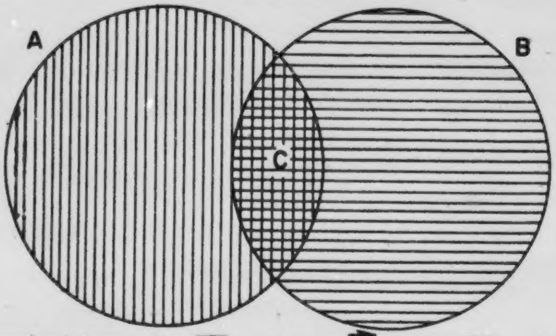
The intersection of $\overline{AB}$ and $\overline{PQ}$ is the set whose only member is the point S.

We write $\overline{AB} \cap \overline{PQ} = \{ S \}$

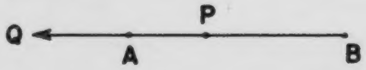
The intersection of the interior of the two circles A and B is the set of points in the shaded interior in which the letter C is placed.

We write

$$\text{Interior of } A \cap \text{Interior of } B = \text{Interior of Region } C$$



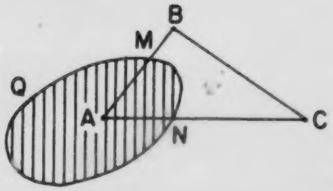
The intersection of $\overline{AB}$ with $\overrightarrow{PQ}$ is $\overline{AP}$. We write $\overline{AB} \cap \overrightarrow{PQ} = \overline{AP}$. It may be observed that this is an abbreviated form for: The intersection of the set of points of $\overline{AB}$ and the set of points of $\overrightarrow{PQ}$ is the set of points of $\overline{AP}$. This rather long and difficult statement should not be required of the pupils but the precise meaning of this and similar statements concerning intersections (and unions) may be helpful to the teacher.



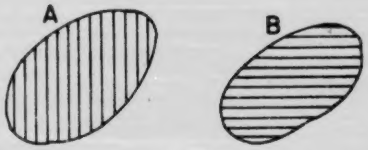
The intersection of the oval region Q and the triangle ABC is $\overline{AM}$ and $\overline{AN}$. (Recall that $\triangle ABC$ does not include its interior.) We write

$$\triangle ABC \cap Q = \overline{AM} \cup \overline{AN}.$$

Again observe that this is the abbreviated statement for: The intersection of the set of points of oval region Q with the set of points of the triangle ABC is the set of points of $\overline{AM}$ and $\overline{AN}$.

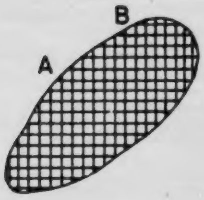


Union of Sets. The union of two sets is a set consisting of the members which are either in one or both of the given sets. Just as we have made Venn Diagrams to illustrate the intersections of sets, we can make diagrams to illustrate the union of sets. The next set of diagrams illustrate union of sets A and B which bear the same relation to each other as the sets A and B used in the intersection diagrams.



$A \cup B$

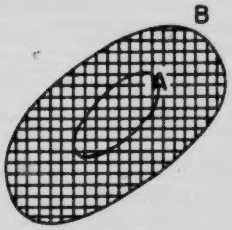
If A and B are disjoint, then their intersection is empty, and their union consists of the sets A and B. The union of A and B is shaded. They have no members in common.



$A \cup B$

If A = B, then $A \cup B = A$

or $A \cup B = B$

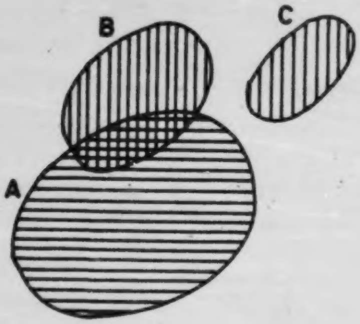


$A \cup B$

If A is a proper subset of B, then

$A \cup B = B$

The following diagrams illustrate both intersections and unions in each diagram.

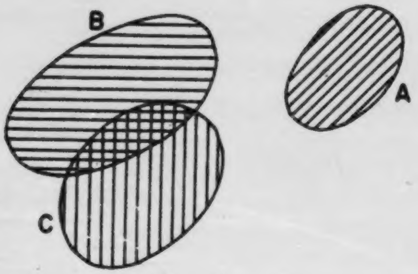


$A \cap (B \cup C)$.

$B \cup C$ is lined vertically.

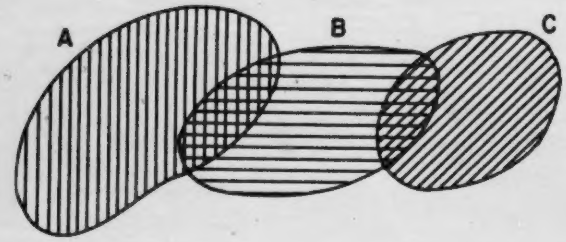
A is lined horizontally.

$A \cap (B \cup C)$ is cross-hatched.



$A \cup (B \cap C)$. A is lined slantingly.

$B \cap C$ is cross-hatched. The union consists of both the lined portion A, and the cross-hatched portion $B \cap C$.



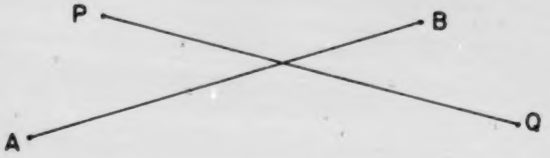
$(B \cap A) \cup (B \cap C)$ or $B \cap (A \cup C)$.

$B \cap A$ is crossed with vertical and horizontal lines.

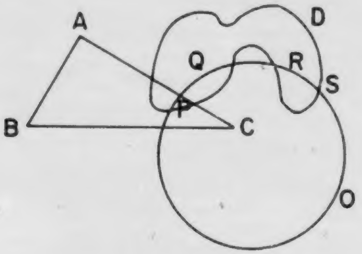
$B \cap C$ is crossed with horizontal and slanting lines.

$(B \cap A) \cup (B \cap C)$ consists of both the portions $B \cap A$ and $B \cap C$.

In the following examples of unions of sets, the drawings are the representations of geometric figures. The sets under consideration will, therefore, consist of points of segments, of circles, of polygons, of the interiors of closed curves, etc. As in the discussion on the distinction between the closed curves of Venn diagrams and actual sets represented by geometric figures, care will need to be observed with the pupils.



The union of $\overline{AB}$ and $\overline{PQ}$ is the two segments $\overline{AB}$ and $\overline{PQ}$; that is, the points that are points of $\overline{AB}$ or $\overline{PQ}$ or of both consist of the points of the union of $\overline{AB}$ and $\overline{PQ}$.



Consider the two sets of points:

Set M = Vertices of $\triangle$ ABC

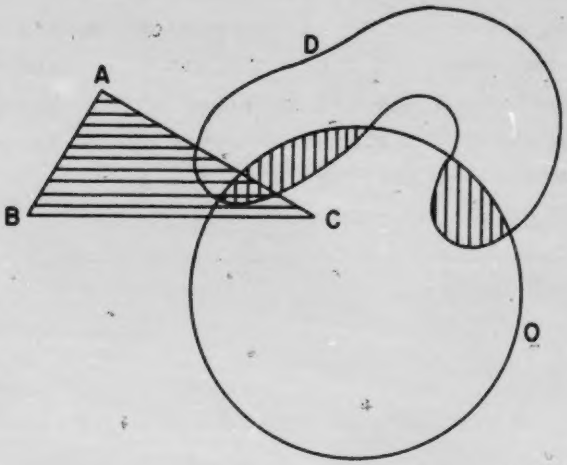
Set K = the intersection of the points of circle O and the closed curve D

Set M = {A, B, C}

Set K = {P, Q, R, S}

$M \cup K = \{A, B, C, P, Q, R, S\}$

Now consider this drawing and look at the two sets of points:



Set X = Set of points in the interior of △ ABC.

Set Y = Set of points in the intersection of the interiors of circle O and the closed curve D.

Set X is the interior of the region lined horizontally.

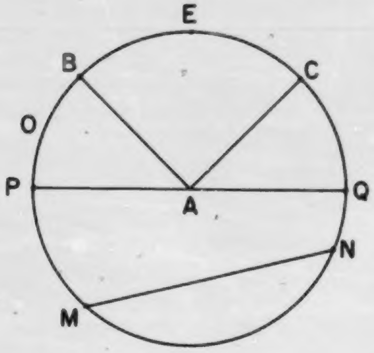
Set Y is the interior of the regions lined vertically.

X ∪ Y consists of the points in the interior lined horizontally, in the interior lined vertically, and in the interior that is lined both horizontally and vertically. In this example it is difficult to describe the union of the two sets in a statement. The vertical and horizontal lining will indicate a way that the pupils may use. Emphasis is upon the selection of the correct interiors, segments, points, etc., and not upon a precise description unless it is fairly easy for the pupils to formulate. Emphasis should not be on language statements to the extent that the central ideas become lost in a linguistic maze.

Circles. A circle is a set of points of the same plane such that all points are equally distant from one point of the plane which is called the center.

The above definition of circle is not in conflict with the alternative statement which defines a circle as a plane simple closed curve all of whose points are equally distant from a point of the plane called the center.

Either definition focuses attention on the circle being a set of points. If one draws a representation of a circle in the usual way, the curve which we see on the paper represents the circle. Care should be taken to distinguish between a circle and its interior. A circle with its interior is called a circular region. In the drawing shown below, A marks the center of the circle. A circle is named by one capital letter written on or near the circle, e.g., O in this drawing.



The line segments AB and AC are radii. These line segments are congruent to each other. Their lengths are the same. They have a common endpoint, A, but the other endpoints are different points of the circle. The lengths of all radii of the circle are the same.

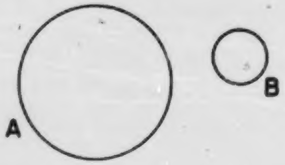
The line segment PQ is a diameter. Any line segment which contains the center of a circle and has points of the circle for its endpoints is a diameter. As is true for radii, all diameters of the circle are congruent to each other but they are not the same diameter. All diameters of a circle have the same length.

The segment MN is called a chord of the circle. Note that its endpoints, M and N, are points of the circle.

A part of a circle is a subset of the circle. It is a set of points that are points of the circle. The set of points (or part of the circle) that you would trace in going from B to C through E is called an arc of the circle and is named arc BEC and written $\overset{\frown}{BEC}$. It could be written $\overset{\frown}{CEB}$ and read arc CEB since these two sets of points are the same set. We could follow a path on the circle from B to C by going through point M (or P, or N, or Q) since these would indicate the same path. This arc could be written in either of these ways: $\overset{\frown}{BPC}$, $\overset{\frown}{BMC}$, $\overset{\frown}{BNC}$, $\overset{\frown}{BQC}$.

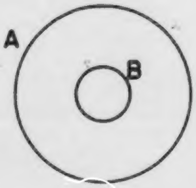
If two circles have the same center, they are called concentric circles. Clearly, two concentric circles whose radii have the same length are the same circle since they are the same set of points.

Two distinct circles may have either no point of intersection, one point of intersection, or two points of intersection. These situations are illustrated in the following drawings and their intersections are stated in "set language."



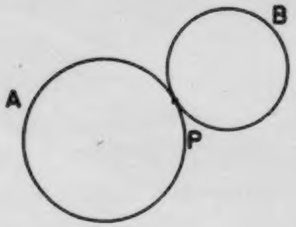
$A \cap B = \{\quad\}$

(1)



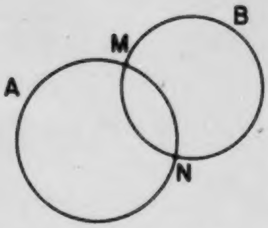
$A \cap B = \{\quad\}$

(2)



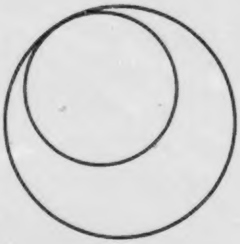
$A \cap B = \{P\}$

(3)

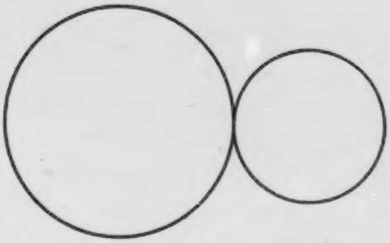


$A \cap B = \{M, N\}$

(4)



(5)



(6)

The drawings (5) and (6) represent circles which are tangent to each other; i.e., their intersection consists of one point.

TEACHING PROCEDURES

This unit differs from many of the preceding units in the sixth grade relative to the use of exploratory passages for the pupils. This is particularly true in the sections devoted to constructions. Many passages include less exploration and more exposition. This is done deliberately, we hope wisely, to obtain some information on the ability of pupils now completing the sixth grade, to read understandingly fairly long passages and follow carefully sets of instructions. A second reason for de-emphasizing the exploratory nature of the presentation is due to the fact that many parts of the unit are a review of previous concepts which were developed initially with the pupils by the exploratory method.

The comments to the teacher in this section TEACHING PROCEDURES are intentionally brief. The explanations to the pupils that precede each set of exercises and the information and examples set forth in the MATHEMATICAL BACKGROUND are thought to be sufficient to provide the teacher with the needed material for teaching the unit with understanding and confidence.

RECOGNIZING SETS

Review with the pupils the concepts in Concept of Sets, Chapter 1, Grade 4. Ask them to name sets of objects, numbers, letters, etc., to emphasize the difference in meaning of a set and the set. A set is indefinite, but the set must be described so that the pupils are certain what the members of the set are and if any object is mentioned, they can tell whether it is a member of the set. You might use such examples as the following to help establish the distinction between a set and the set.

"A set of books" does not name a specific set. We have no way of knowing what books are meant. But, "the set of books on the top shelf of the bookcase in our schoolroom" is a specific set. The pupils can tell what books are in the set. They can also tell whether any book that you mention belongs to the set.

"A set of letters in the alphabet" is not a specific set. It does not tell us how many letters are in the set or what letters are meant. But, "the set of the first ten letters of the alphabet" is a specific set. We can write, or say, these letters if we wish. Have the pupils write the set and denote it by T:

$$T = \{a, b, c, d, e, f, g, h, i, j\}$$

Ask the pupils to tell which of the following sets are described well enough for them to know what the members of the set are:

1. A set of numbers. (They cannot tell)
2. A set of blocks. (They cannot tell)
3. The set of whole numbers which are less than 6. (0, 1, 2, 3, 4, 5)
4. A set of states. (They cannot tell)
5. The set of states of the United States whose names begin with X. (They can tell, it is the empty set.)
6. A set of shoes. (They cannot tell)
7. The set of days of the week. (Sunday, Monday, etc.)
8. A set of pupils. (They cannot tell)
9. The set of pupils in your classroom. (They can tell)
10. The set of whole numbers less than 100 which are divisible by 5. (0, 5, 10, 15, ... 95)

Study the material with the pupils in their text on Recognizing Sets; then proceed to Exercise Set 1.

Chapter 9

SETS AND CIRCLES

RECOGNIZING SETS

If you saw a great many birds flying overhead and wanted to tell someone about what you saw, how would you tell them? Would you say "I saw a lot of birds" or "I saw a flock of birds" or "I saw a group of birds"? You would be more likely to say "I saw a flock of birds". But, if you saw bees instead of birds, you might say "I saw a swarm of bees", or "I saw a bunch of bees".

A man who has a big ranch in Texas may have many cattle and many horses. If he were to talk to you about these, he might say he had a "herd of cattle" and a "drove of horses."

You see, in these first two paragraphs we have used seven different words in the same way. These words are underlined. Is each a word used to refer to a collection of things? Now, you may remember that we decided to use just one word to speak about a collection of things. This word was the word set. If we do this we would say "A set of birds", A set of bees", "A set of cattle", "A set of horses".

If someone tells you that she is thinking about a set of dishes, does this tell you very much about the dishes in the set? Do you know what color they are? How many are in the set? Are they cups, or saucers, or plates, or some of each of these? The answer to most of these questions is "No". So, when someone says "a set of dishes" she is not really telling you very much about them. But if she were to say "the dishes you see in the window at a certain china store", she would be talking about a certain set. You cannot know what dishes are meant by "a set of dishes". But you can know what dishes are meant by "the set of dishes you see in the window at a certain china store".

Do you know what boys are meant if we say "a set of boys"? (no) Do you know what boys are meant if we say "the set of boys in your class whose names are Tom, Dick, or Harry"? (yes)

In each of the first five exercises below, write on your paper a mathematical sentence for the set. Notice this example first.

Example: The set of streets that cross at First Street and Main Street. S = {First Street, Main Street}.

1. The set of odd whole numbers which are less than 10.

   T = {1, 3, 5, 7, 9}

2. The set of even numbers which are less than twenty and greater than 8.

   E = {10, 12, 14, 16, 18}

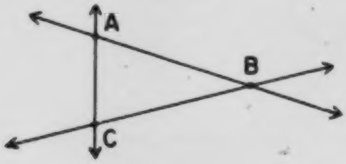
3. (a) The set of whole numbers which are factors of 30.

   W = {1, 2, 3, 5, 6, 10, 15, 30}

   (b) The set of prime numbers which are factors of 30.

   Q = {2, 3, 5}

4. The set of letters which label the points of intersection of the three line segments shown in this drawing.



P = { A, B, C }

5. The names of the states in the United States whose names begin with H.

Y = { Hawaii }

In some of the next five sets you are not told enough about the sets so you can write a mathematical sentence for them. Which ones are they? (6, 10)

6. Set of stamps.

7. Set of books on a certain shelf of the bookshelf in your classroom.

8. Set of letters in the first half of the alphabet.

9. Set of whole numbers less than 119 whose numerals have 1 for the first (leftmost) digit.

10. Set of cards.

Exercise Set 1

For each of the first five exercises below, write a mathematical sentence on your paper for each set.

1. The set of counting numbers less than 40 which are multiples of 5. $A = \{5, 10, 15, 20, 25, 30, 35\}$

2. The set of prime numbers which are greater than 17 and less than 29. $B = \{19, 23\}$

3. The set of whole numbers which are factors of 60. $C = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

4. The set of whole numbers less than 480 which are multiples of 60. $D = \{60, 120, 180, 240, 300, 360, 420\}$

5. The set of names of the states of the U.S.A. whose first letter is Z. $E = \{\ \}$

In some of the next six exercises you are not told enough about the sets so you can write a sentence for them. Which ones are they? (6, 8)

6. Set of letters.

7. Set of whole numbers which are greater than 10 and less than 11.

8. Set of points which are the vertices of a triangle.

9. Set of members of this set: {1, 4, +, , $\mathcal{L}$ }.

10. Set of titles of the books that are on your teacher's desk.

11. Set of names of the presidents of the U.S.A. since 1900.

12. Describe in words the set { }.
(This is the symbol for an empty set. An empty set has no members.)

13. Would the dishes your mother used last Thanksgiving and your father's best hammer form a set? (Yes) Why? (Because you are describing a particular set of dishes and a particular hammer.)

14. Would the men in your town that are at least six feet tall and weigh no more than one pound form a set? (Yes) Why? (This set of men would from the empty set.)

15. Name the members of the set of (Answers will vary.)

(a) red-haired boys in your class

(b) blue-eyed boys in your class

(c) red-haired blue-eyed boys in your class.

INTERSECTION OF SETS

You might begin the study of this section with the pupils by writing some pairs of sets on the blackboard. The sets should not have a large number of members. Have some pairs which have no members in common, some pairs which have a small number of members in common and some pairs which have all members in common.

Let us look at the two sets (write them on the board):

$$A = \{3, 6, 9, 12\} \qquad B = \{9, 12, 15, 18\}$$

Questions you might ask. Are there members in one set that are in the other set also? Or, you may ask the same question this way: Do the two sets have members in common? Yes, the numbers 9 and 12 are in both sets. Then the set whose members are the numbers 9 and 12 is the intersection set of A and B. We write this

$$A \cap B = \{9, 12\}$$

Have the pupils read this, "the intersection of sets A and B is the set whose members are 9 and 12."

Use such examples as the following to sharpen the pupils' recognition of intersection of sets and have them write and read the statements that describe the intersection.

(1) $C = \{c, a, t, s\}$
$D = \{o, a, t, s\}$ $\qquad C \cap D = \{a, t, s\}$

(2) $R = \{m, r, s\}$
$T = \{m, r\}$ $\qquad R \cap T = \{m, r\}$

(3) $X = \{1, 7, 11, 9, 15, 13, 5, 2\}$
$Y = \{13, 7, 1, 9, 11, 5, 15, 2\}$
$X \cap Y = \{1, 7, 11, 9, 15, 13, 5, 2\}$

(4) $\text{Set } 1 = \{a, b, c, d, e, f\}$
$\text{Set } 2 = \{a, b, c\}$
$\text{Set } 1 \cap \text{Set } 2 = \{a, b, c\}$

Some sets should be described in words. Help the pupils describe the intersection. These will not be written in the same form as used on the preceding page. The following examples may be helpful:

(1) If set M is the set of all whole numbers less than 100 and set N is the set of all whole numbers less than 200, describe the intersection of M and N. (The intersection of M and N is the set of whole numbers from 0 through 99, or Set M)

(2) Set V is the set of pupils in your class who are members of the school chorus. Set W is the set of pupils in your class who are members of the school band. Describe the intersection of V and W. (The intersection may be found by writing the names of the pupils that are common to both sets V and W. If there are no pupils common to both sets, then $V \cap W = \{\quad\}$.)

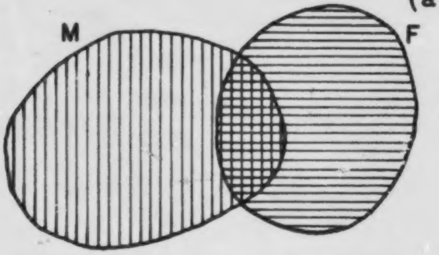
(3) Set P is the set of all elementary teachers in your town (city, county) and Set Q is the set of all sixth-grade teachers in your town (city, county). Describe the intersection of sets P and Q. (The intersection of sets P and Q is the set Q since all teachers in set Q are also in set P.)

(4) If one set consists of the students in your school who are studying mathematics and another set consists of all students in your school who are studying French, might these two sets have an intersection set which is not the empty set? (Yes, there may be students in your school who are studying both mathematics and French and these would be in the intersection set. If there are no students in your school who are studying both mathematics and French, then the intersection of the two sets is the empty set.)

After practice on such problems as (1) through (4) above, the Venn Diagrams for intersections may be introduced. Use the information in (4) in building a Venn Diagram.

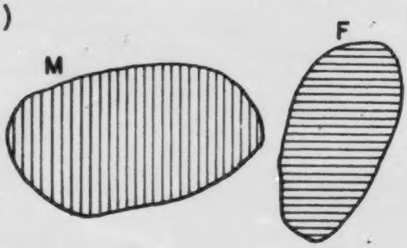
__Steps__ __in__ __building__ __the__ __diagram__: Let the oval region M, lined with vertical lines, represent the students in the school who are studying mathematics. Now, before we draw a region to represent the students who are studying French, we must know which one of the following situations exists:

(a)



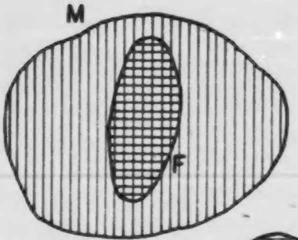
(a). Some but not all of the students are studying both mathematics and French. The Venn diagram for this situation is diagram (a). The two regions overlap.

(b)



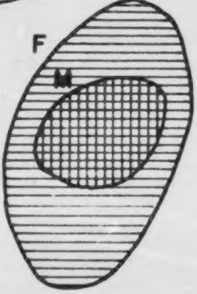
(b) There are no students taking both mathematics and French. This is Venn diagram (b) The two regions do not overlap.

(c)



(c) All students who are taking French are taking mathematics also. This is Venn diagram (c). The F region is within the M region.

(d)



(d) All students who are taking mathematics are taking French also. This is Venn diagram (d). The M region is within the F region.

The sizes and shapes of the regions are __not__ important. The Venn diagrams serve only one purpose, namely to represent diagrammatically the information conveyed by the sentences.

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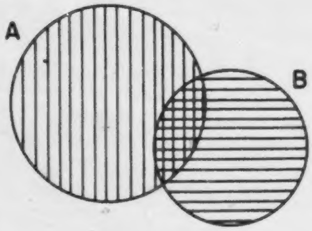
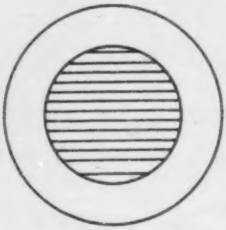
INTERSECTION OF SETS

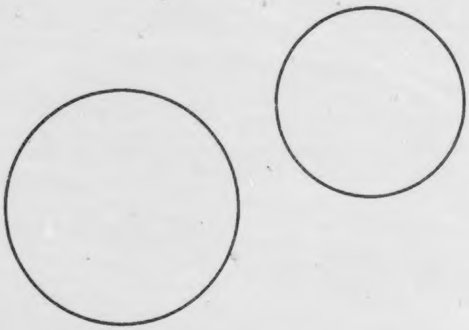
The two circular regions below are drawn so that they overlap. The circular region A is lined vertically. The circular region B is lined horizontally. The intersection is cross-hatched. The union of the interior of the two circular regions consists of three portions: The portion that is lined vertically, the portion that is lined horizontally, and the portion that is cross-hatched. Remember that union of two sets of points includes all the points that are points of either one of the sets or points of both of the sets.



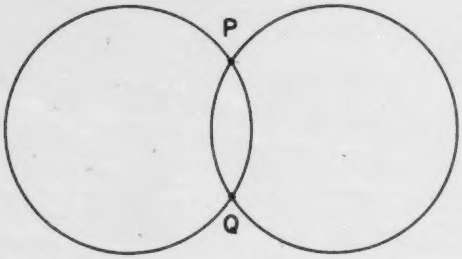
In the drawing below the interiors of the two circular regions intersect in the area that is shaded. Remember that the intersection of two sets is the set of elements that are in both sets. The members in the sets in this figure are the points of the two circular regions. The points of the smaller regions are in the larger region, so the intersection set is the set of points of the smaller region.



In the next drawing the union of the two circles consists of the two circles. Or you may say that the union of the sets of points on the two circles consists of all the points on the two circles. Draw the circles on your paper and color the union set with a colored crayon.



In the next drawing there are just two points that are on both circles. Name them P and Q. Are the points P and Q members of the intersection set of the interiors of the two circles? (no) Are they members of the union set of the two circles? (yes) Are they the only members of the intersection set of the two circles? (yes)



Do you remember what we mean by the intersection of two sets? Perhaps an example will help recall it.

Look at these two sets:

$$A = \{c, a, n\}$$

$$B = \{b, a, t\}$$

The letter a is in both sets. We call this the intersection of sets A and B. We write this

$$A \cap B = \{a\}$$

The intersection of two sets is the set of members which are in both of the given sets. In the two sets above, a is the only letter that is in both sets.

What will be the intersection of two sets if there is no member in one set that is also in the other? The intersection is the empty set, $\{\ \}$.

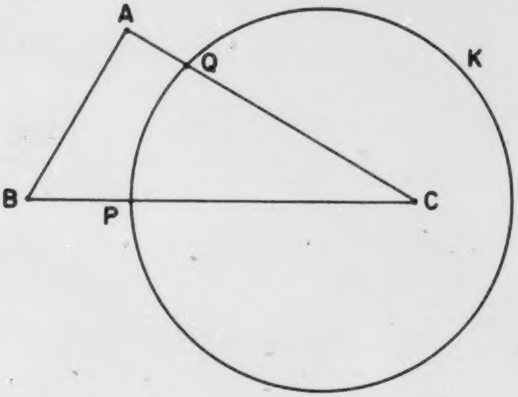
Examples of intersections of sets:

1. $S = \{2, 3, 4, 5\}$ $T = \{1, 2, 3\}$ $S \cap T = \{2, 3\}$
2. $A = \{a, b, c\}$ $B = \{c, d, e\}$ $A \cap B = \{c\}$
3. $M = \{a, e, i, o, u\}$ N = set of consonants in the alphabet

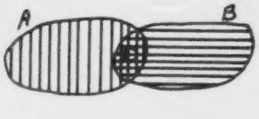
$$M \cap N = \{\ \}.$$

4. In the drawing below are two geometric figures. One is a triangle and the other is a circle. There are two points, P and Q, which are on both the circle and on the triangle. So P and Q are the points of intersection. We write this

$$(\triangle ABC) \cap (\text{Circle } K) = \{P, Q\}$$



5. In a certain school some of the pupils are studying English and some are studying History. There are some, but not all, of these pupils that are studying both English and History. Draw a Venn diagram to represent this situation.



The vertically lined region A may represent the pupils studying English
The horizontally lined region B may represent the pupils studying History.
The cross hatched region C may represent the pupil studying both English and History.

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BRAINTWISTER

6. In another school there are 47 pupils. The subjects they study are History, English, and Science.

4 students study all three subjects

5 students study History and English but not Science

6 students study History and Science but not English

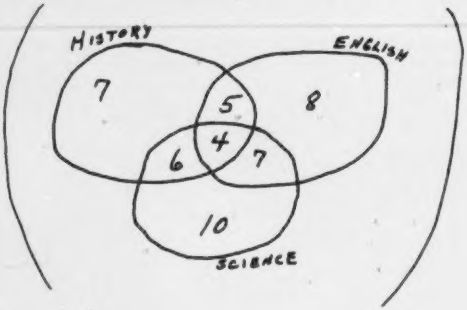
7 students study English and Science but not History

7 students study History only

8 students study English only

10 students study Science only

Draw a Venn diagram to illustrate this situation. You will need to use 3 regions.



Exercise Set 2

1. If $S = \{a, b, 3, 5\}$, $T = \{□, 3, c, d\}$, $W = \{5\}$, and $E = \{\ \}$, then find

a. $S \cap T = \{3\}$ c. $T \cap E = \{\}$ e. $T \cap W = \{\}$

b. $S \cap W = \{5)$ d. $W \cap E = \{\}$ f. $S \cap E = \{\}$

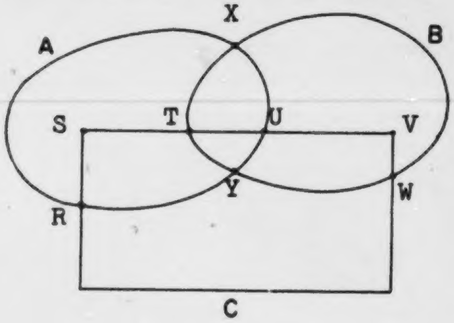
2. If $A = \{4, 3, 1, 2\}$, $C = \{3, 4, 2, 1\}$.
$B = \{a, 4, 7\}$, and $D = \{□, 5\}$,

then find

a. $A \cap C = \{1, 2, 3, 4\}$ c. $C \cap D = \{\}$

b. $A \cap B = \{4\}$ d. $A \cap F$, if $F = B \cap C$.
$F = \{4\}$, $A \cap F = \{4\}$

3. Trace a drawing similar to the one pictured below. The simple closed curves are labeled A, B, and C.
(Note: Remember how simple closed curves are defined.)

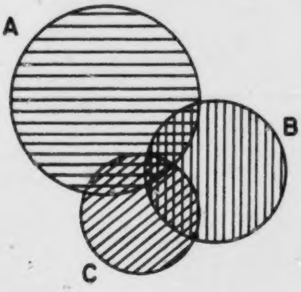


Find

a. $A \cap B = \{X, Y\}$ c. $B \cap C = \{T, W\}$

b. $A \cap C = \{R, U\}$ d. $(A \cap B) \cap C = \{\}$

4. Draw circular regions A, B, C as shown. Draw horizontal segments in the region A as shown. Then draw vertical segments in the region B and slanting segments in the region C.



Describe how each of the following is lined.

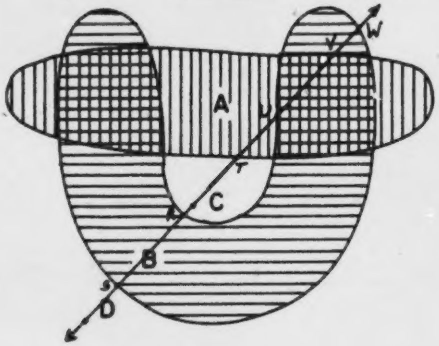
a. $A \cap B$ (Lined with vertical and horizontal lines)

b. $B \cap C$ (Lined with vertical and slanting lines.)

c. $A \cap C$ (Lined with horizontal and slanting lines.)

d. $A \cap (B \cap C)$ (Lined with horizontal, vertical, and slanting lines)

5. Look at the oval-shaped region A lined with vertical lines, the horseshoe-shaped region B lined with horizontal lines, and the line CD with some points labeled on it. Now describe each of the following intersection sets:



a. $A \cap B$ (Lined with vertical and horizontal lines.)

b. $A \cap \overrightarrow{CD} = \{\ \}$

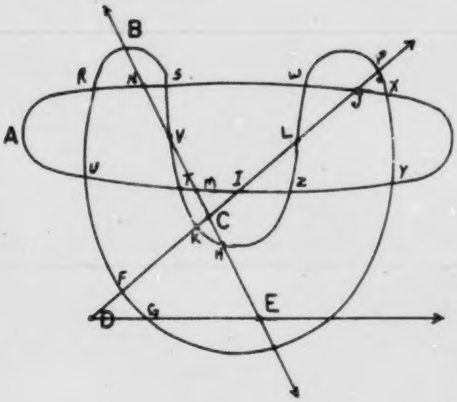
c. $A \cap \overrightarrow{DC}$ (The intersection is the set of points of $\overline{TV}$.)

d. $B \cap \overrightarrow{CD}$ (The intersection is the set of points of $\overline{RS}$.)

e. $B \cap \overleftrightarrow{CD}$ (The intersection is the sets of points of $\overline{SR}$ and $\overline{UW}$.)

f. $A \cap \overline{CD}$ $= \{\ \}$

6.



Look at the oval-shaped curve A, the horseshoe-shaped curve B, the ∠PDE, the line BE and the points that are labeled.

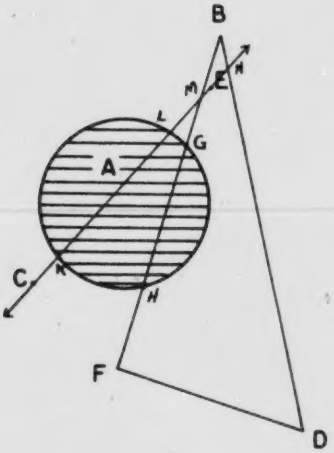
Describe each of the following intersection sets.

a. A $\cap$ B = {U,R,S,T,W,X,Y,Z}
b. B $\cap$ $\overrightarrow{DC}$ = {F,K,L,P}
c. A $\cap$ $\overrightarrow{DC}$ = {I,J}

d. A $\cap$ $\overleftrightarrow{EC}$ = {M,N}
e. $\overline{CD}$ $\cap$ ∠ DEC = {C,D}
f. A $\cap$ ∠ CDE = {I,J}

g. A $\cap$ {E} = { }
h. A $\cap$ {C, D, E} = { }
i. (A $\cap$ B) $\cap$ A = {U,R,S,T,W,X,Y,Z}

7.



Look at the circular region A, the △ BDF, the line CE, and the labeled points. Describe each of the following intersection sets.

a. A $\cap$ △ BDF
(The intersection is the set of points of $\overline{GH}$)
b. A $\cap$ $\overleftrightarrow{CE}$
(The intersection is the set of points of $\overline{KL}$)

c. △ BDF $\cap$ $\overline{CE}$ {M}
d. △ BDF $\cap$ $\overrightarrow{CE}$ {M,N}

e. △ BDF $\cap$ $\overrightarrow{EC}$ {M}
f. △ BDF $\cap$ {E, C} { }

UNION OF SETS

The following is a suggested procedure for the development of this topic. You should expect answers from the pupils instead of supplying them. Allow the pupils plenty of time to see the members that belong to the union. As you write sets on the board, let the pupils write the same sets on their paper. Help the pupils in writing the sets correctly.

Recall what is meant by the union of two or more sets. We will do this by some examples:

Example 1

Suppose set $A = \{1, 2, 3, 4, 5\}$
and set $B = \{4, 5, 6, 7, 8\}$

The union of set A and set B is $\{1, 2, 3, 4, 5, 6, 7, 8\}$
We write this: $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
Notice that 4 and 5 are in both sets but each appears only once in the union.

Example 2

Suppose one committee for your class party has James, Robert and Marie as its members. Call this committee R (R for refreshments).

Then R = {James, Robert, Marie}

Another committee has Lloyd, Marie, and Mildred as its members. Call this committee D (D for decorations).

Then D = {Lloyd, Marie, Mildred}

What is the set of students in your class that are committee members? This is the set: James, Robert, Marie, Lloyd, Mildred. It is called the union of the two sets R and D. We write this,

R ∪ D = {James, Robert, Marie, Lloyd, Mildred}

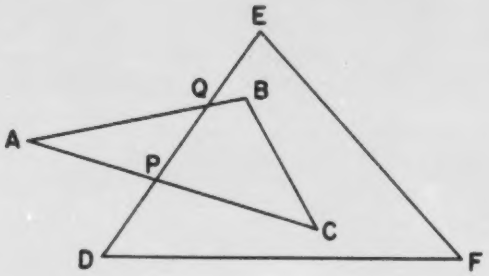
These two examples should help us recognize the union of any two sets. Let us say this in three easy sentences:

(1) The union of any two sets is a set.

(2) A member of just one of the two sets is a member of their union.

(3) Also, a member of both of the two sets is a member of their union.

Now let us use some pictures of geometric figures and select some sets of points from them and find their unions.



Look at the two triangles ABC and DEF in the picture. What is the set of points that is the union set of the two triangles? The union set consists of all the points of triangle ABC, of triangle DEF, and of points that are points of both triangles.

Hence:

$$\triangle ABC \cup \triangle DEF = \{\text{All points of both } \triangle\text{'s}\}$$

Or, we may say, the union is the set of points on the segments AB, BC, AC, DE, EF, and DF. Are the points P and Q in the union? Yes. Is each point P and Q counted twice in the union? No, only once. Are the points P and Q the intersection set of the two triangles? Yes.

We may write

$$\triangle ABC \cap \triangle DEF = \{P, Q\}$$

UNION OF SETS

A class was making arrangements for a party. Two committees were set up. One was the committee for refreshments. Its members were Mary, Joan, and Bill. Call this set R.

We can write

$$R = \{\text{Mary, Joan, Bill}\}$$

The other committee was the committee for decorations. Its members were Harry, Bill, and Henry. Call this set D.

We can write

$$D = \{\text{Harry, Bill, Henry}\}$$

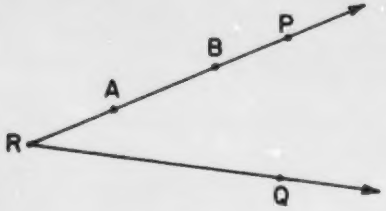
The pupils who are on committees are Mary, Joan, Bill, Harry, and Henry. This set is called the union of the two sets R and D. Its members are the members which are in either R or D or both.

We write this

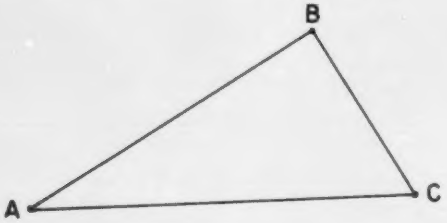
$$R \cup D = \{\text{Mary, Joan, Bill, Harry, Henry}\}$$

Each name appears only once in the set $R \cup D$. Bill is on both committees and each of the other pupils is on one of the committees.

Let us look at another example of the union of two sets. Look at the geometric figures in the next drawing. Think of the line segment AB and the angle PRQ. Is the set of points of $\overline{AB}$ contained in the set of points of ∠PRQ? (yes)



What is the union of the set of points of $\overline{AB}$ and the set of points of the angle PRQ? (∠PRQ) Do you see that the points of $\overline{AB}$ are also points of the angle PRQ? (yes) The points of $\overline{AB}$ are points of both of the sets of points in the drawing. All the other points in the drawing are points of ∠PRQ. Hence, the union of the two sets $\overline{AB}$ and ∠PRQ is the ∠PRQ.



Look at the triangle ABC in the figure shown above. It consists of the points of the three line segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$. We can say that the union of the three line segments $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ is the triangle ABC.

We write:

$$(\overline{AB} \cup \overline{BC}) \cup \overline{CA} = \triangle ABC.$$

Exercise Set 3

1. If $A = \{a, b, c, d\}$

   $B = \{c, 1, 2\ \triangle\}$

   $E = \{\ \}$

   Then what is

   $A \cap B = \{c\}$

   a. $A \cup B$ $\{a, b, c, d, 1, 2, \triangle\}$

   b. $A \cup E$ $\{a, b, c, d\}$

   c. $A \cup (A \cap B)$ $\{a, b, c, d\}$

   d. $B \cup (A \cap B)$ $\{c, 1, 2, \triangle\}$

2. If A is the set of the five most commonly used vowels, and B is the set of the first 10 letters of the alphabet, and C is the set of the last 10 letters of the alphabet, then what is

   $A = \{a, e, i, o, u\}$
   $B = \{a, b, c, d, e, f, g, h, i, j\}$
   $C = \{q, r, s, t, u, v, w, x, y, z\}$

   a. $A \cup B$
   $\{a, b, c, d, e, f, g, h, i, j, o, u\}$

   b. $B \cup C$
   $\{a, b, c, d, e, f, g, h, i, j, q, r, s, t, u, v, w, x, y, z\}$

   c. $A \cup \{w, y\}$
   $\{a, e, i, o, u, w, y\}$

   d. $(A \cup B) \cap C$ $\{u\}$

   e. $(A \cap C) \cup (B \cap C)$ $\{u\}$

   f. $\{u, w, y\} \cup (A \cap B)$
   $\{a, e, i, u, w, y\}$

3. If A is the set of students in your class, and B is the set of blue-eyed students in your class, and C is the set of red-haired students in your class, then what is described in each of the following:

   a. $A \cup B$ (Student who are in the class and blue eyed.)

   b. $A \cup C$ (Student who are in the class and red-haired.)

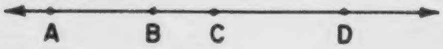
   c. $C \cup (A \cap B)$
   (All red-haired student and all blue-eyed student in the class.)

   d. $A \cup (B \cup C)$
   (All student in the class)

   e. $(A \cup B) \cup C$
   (All students in the class)

   f. $A \cup (B \cap C)$
   (All student in the class)

4.



Copy each statement a, b, c, d, and complete it so that it will be true.

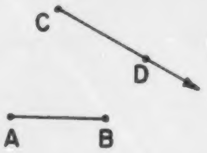
a. $\overline{AC} \cup \overline{BC} = \overline{AC}$

b. $\overrightarrow{AB} \cup \overrightarrow{BC} = \overrightarrow{AC}$

c. $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$

d. $\overleftrightarrow{AC} \cup \overleftrightarrow{BC} = \overleftrightarrow{AC}$

5. Copy each statement a, b, c, and complete it so that it will be true.

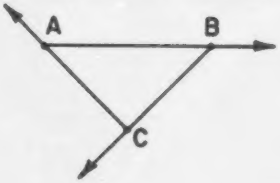


a. $\{A\} \cup \{C\} = \{A, C\}$

b. $(\overline{CD} \cup \overline{BC}) \cup \overline{BD} = \triangle BCD$

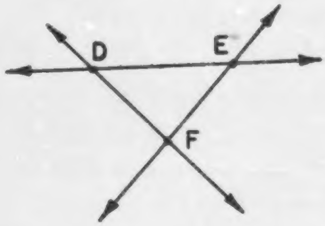
c. $(\overline{CD} \cup \overline{BC}) \cup (\overline{BD} \cup \overline{AB})$ $\triangle BCD \cup \overline{AB}$

6. Mark points A, B, and C (not in the same line) on your paper. Draw $(\overrightarrow{AB} \cup \overrightarrow{BC}) \cup \overrightarrow{CA}$. Is the figure a triangle?
(No, it is not even a simple closed curve.)



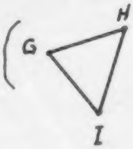
Is a triangle a union of three rays?
(No. The union of three rays is not even a simple closed curve.)

7. Mark points D, E, and F (not in the same line) on your paper. Draw $(\angle DEF \cup \angle EFD) \cup \angle DEF$. Is the figure a triangle? (No. It is not even a simple closed curve.)



8. Mark points G, H, and I (not in the same line) on your paper. Draw $(\overline{GH} \cup \overline{HI}) \cup \overline{IG}$.

a. Is the figure a triangle? (yes)



b. Is a triangle a union of three line segments? (yes)

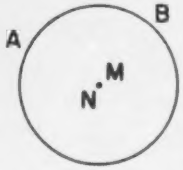
c. Is the union of three line segments always a triangle? (No. The union of the three line segments might not be a simple closed curve.)

d. What is a triangle? (A triangle is a simple closed curve which is the union of three line segments.)

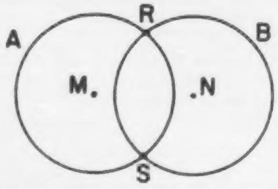
CIRCLES

Let us begin by drawing circles (really pictures of circles) on the chalkboard. (Or, have the pupils use compasses to draw on paper). Pupils may need review in use of compass and in use of chalk and string for drawing on their paper and chalkboard. Draw some circles on thin paper so that they may be compared by placing one on top of another. Lead the pupils to recognize that if two circles have a radius of one congruent to a radius of the other, then the two circles are congruent to each other. Also, compare two circles which have a radius of one greater than a radius of the other. Help the pupils recognize that if two circles have radii of different lengths, then the interiors of the two circles are unequal. Which area was the greater measure?

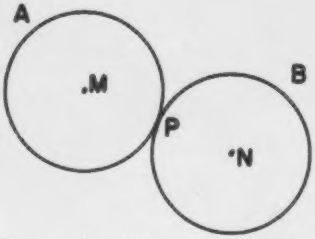
Have the pupils experiment in drawing two congruent circles which are in different relative positions to each other. They should be led to the following conclusions, each of which is illustrated by the associated drawing and stated just below the drawing.



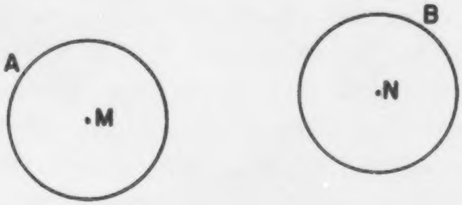
Two congruent circles will coincide or be the same circle if they have the same center. In set language, the intersection of two congruent circles is the set of points on both circles.



Two congruent circles may intersect in two points. In set language, the intersection of two congruent circles may be a set of two points.

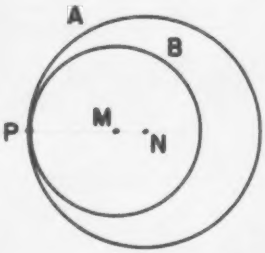


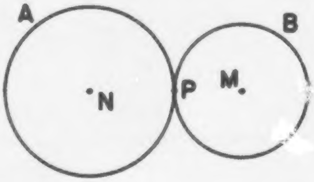
Two congruent circles may intersect in one point. In set language, the intersection of two congruent circles may be one point. (The pupils may wish to say that the circles A and B touch each other at the point P. This is permissible.) Do the three points M, P, N seem to be in a straight line? (yes)

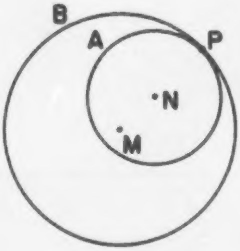


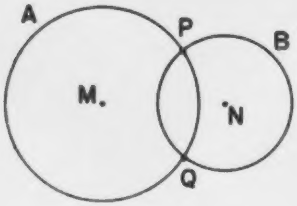
Two congruent circles may not intersect at all. In set language, the intersection of two congruent circles may be the empty set.

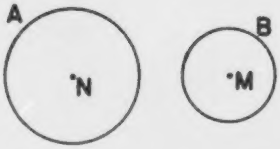
Next, let us examine the intersections of two circles which are not congruent. The pupils should discover the situations shown in the following drawings and make the appropriate statements concerning the intersections.

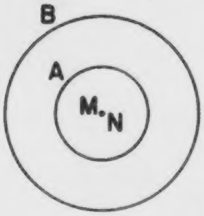






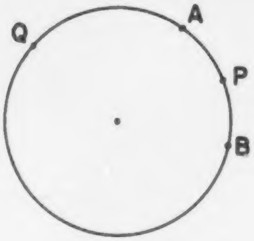






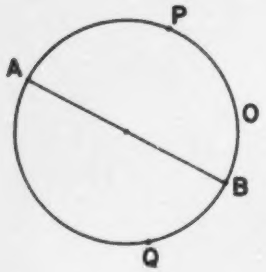
When two circles intersect in just one point, do the centers and the point of intersection always seem to be on a straight line? (Yes)

Let us recall what we mean by an arc of a circle. A part of the circle which can be traced with a pencil without lifting the pencil from the drawing is called an arc of the circle. In the drawing the part of the circle from A to B is an arc. But, does this tell us which arc is meant? Do we mean the arc from A to B in the direction a clock hand moves, or is it the arc from A to B in the direction opposite to that in which a clock hand moves? We cannot tell; it could mean either. So to distinguish the two arcs we will use letters placed along the arcs between A and B and read the arcs as shown below the drawing.



Arc APB, written $\widehat{APB}$, is the shorter arc.

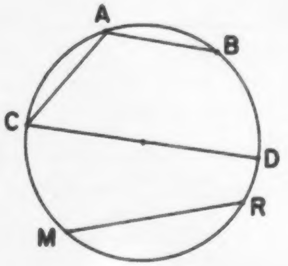
Arc AQB, written $\widehat{AQB}$, is the longer arc.



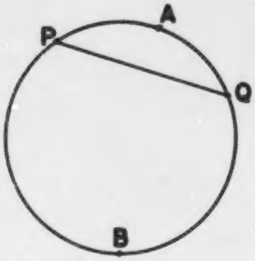
If A and B are the endpoints of a diameter of the circle, then $\widehat{APB}$ is congruent to $\widehat{AQB}$ and each arc is called a semi-circle of the given circle. $\widehat{APB}$ is congruent to $\widehat{AQB}$ and each arc is a semi-circle of the circle O. Will the endpoints of any diameter of a circle divide the circle into two congruent semi-circles? (Yes)

Now, have the pupils draw some circles and on the circles color some arcs which are semi-circles, some which are less than semi-circles; have them read and write the name of each arc, using three letters to name each arc.

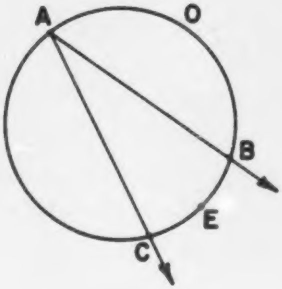
A line segment which has its endpoints points of a circle is called a chord of the circle. In the drawing shown below, $\overline{AB}$, $\overline{AC}$, $\overline{MR}$, are chords of the circle.



Is $\overline{CD}$ a chord of the circle? (Yes, a diameter is a chord. The length of a diameter of a circle is the greatest length that any chord of a circle may have.) The endpoints of a chord of a circle separate the circle into two arcs.



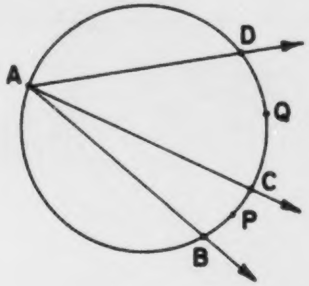
The endpoints P and Q of the chord PQ separate the circle into arcs $\overset{\frown}{PAQ}$ and $\overset{\frown}{PBQ}$.



An angle which has its vertex on a circle and which has two other points in common with the circle is said to be inscribed in the circle and is called an inscribed angle.

The $\angle BAC$ is an inscribed angle. The $\overset{\frown}{BEC}$ is in the interior of the angle. We say $\angle BAC$ intercepts $\overset{\frown}{BEC}$ on the circle O. The letter E is not really needed here since the arc intercepted by an angle is always the arc in the interior of the angle.

In such figures as the following, let the pupils name the chords, inscribed angles, and the arc intercepted by each inscribed angle.



Chords: $\overline{AB}$, $\overline{AC}$, $\overline{AD}$.

Inscribed angles and the arc which each intercepts.

$\angle$ BAC: $\overset{\frown}{BPC}$

$\angle$ BAD: $\overset{\frown}{BCD}$ or $\overset{\frown}{BPD}$ or $\overset{\frown}{BQD}$

$\angle$ CAD: $\overset{\frown}{CQD}$

Now study with the pupils the material on Naming Circles and proceed to Exercise Set 4.

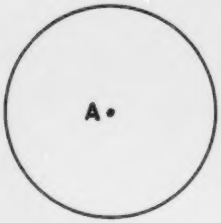
P567

CIRCLES

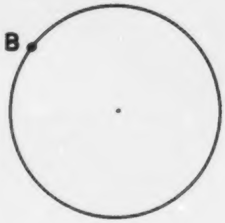
Naming Circles

You know how to draw a picture of a circle by using your compass. Or, if you are working at the chalkboard you will probably use a piece of string and a piece of chalk. The sharp point of the compass that is placed on your paper marks the center of the circle. All the points of the circle are the same distance from its center.

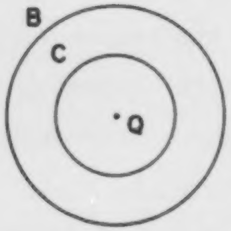
In the drawing below, the point A is the center of the circle. If you wish to speak of this circle you may call it Circle A.



Or, if you like you may mark some point of the circle as in this drawing and speak of the circle as Circle B.

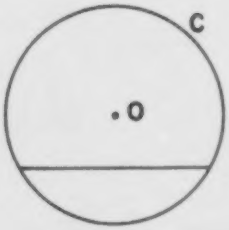


If you have two circles with the same center as in the next drawing, which will be the better way of speaking of them? (Circle B and Circle C)



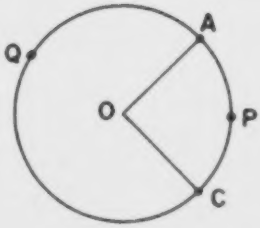
What is the intersection set of the two circles in the figure where the two circles have the same center Q? { } What is the union set of these two circles? (The two circles B and C)

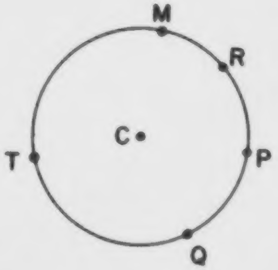
If you draw a line segment which has both of its endpoints on a circle, the line segment is called a *chord* of the circle. Draw a circle on your paper. Then draw a chord of the circle.



How many chords could be drawn *in* a circle? (More than you can count.) Is there more than one chord *of* a circle? (yes) Can you draw a chord of the circle which will pass through the point O? (yes) Can you draw more than one chord which will pass through the point O? (yes) The name for a chord which passes through O is *diameter*.

An arc of a circle is a particular subset of the circle. In the drawing below, the subset of the circle between A and C is called an arc. But there are two subsets of the circle between A and C. One subset is the arc on which we have marked the point P and the other is the subset on which we have marked the point Q. We can tell one from the other if we name one arc APC and the other arc AQC. We write these $\overset{\frown}{APC}$ and $\overset{\frown}{AQC}$. You see that we write the name of the arc in much the same way that we write the name of a line segment. But we need three letters to name an arc while two letters are enough to name a line segment.

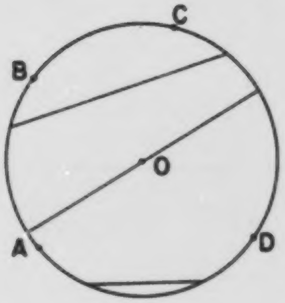




Write on your paper the names of some of the arcs of the circle whose center is C. Remember to use 3 letters in naming an arc. ($\overset{\frown}{TMR}$, $\overset{\frown}{TMP}$, $\overset{\frown}{TMQ}$, $\overset{\frown}{MRP}$, $\overset{\frown}{RPQ}$, $\overset{\frown}{PQT}$, $\overset{\frown}{QTM}$, $\overset{\frown}{TRP}$, etc.)

Exercise Set 4

1.



a. Mark a point O on your paper and draw a circle as shown. Is O a point of the circle? (No)

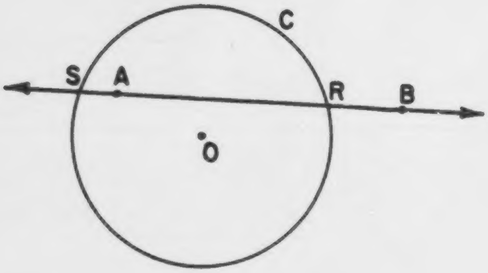
b. Mark points A, B, C, and D on the circle. Name at least three arcs of your circle. ($\overset{\frown}{ABC}$, $\overset{\frown}{BCD}$, $\overset{\frown}{CDA}$, $\overset{\frown}{ACD}$, etc.)

c. Draw a chord of your circle. Draw another chord of your circle. Are the chords the same length? (No)

d. Do you think this circle has a longest chord? (yes) If so, draw it. (Chord passing through O.)

e. Do you think each circle has a shortest chord? (No) If so, draw it. If not, why not? (Because a chord joins two points on a circle and between any two points there are other points.)

2.



a. Mark a point O on your paper and draw a circle as shown. Name your circle C. Is point O in the interior of C? (Yes)

b. Mark a point A in the interior of your circle. Mark a point B which is not in the interior and is also not a point of the circle.

c. Is $\overline{AB}$ in the interior of your circle? (No, not completely.)

d. Is any point of $\overline{AB}$ a point of your circle? (Yes, point R is on $\overline{AB}$ and circle C.)
Is $\overline{AB}$ a part of any circle? (No)

e. $\overline{AB} \cap C =$? $\{R\}$

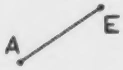
f. $\overleftrightarrow{AB} \cap C =$? $\{R, S\}$

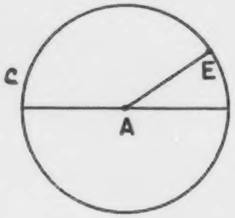
g. $\overrightarrow{AB} \cap C =$? $\{R\}$

h. $\overrightarrow{BA} \cap C =$? $\{R. S\}$

3. If l represents a line and C represents a circle, then the set $l \cap C$ has how many members? Draw enough pictures to illustrate your answer. ($l \cap C$ may have none, one, or two members.)

4.





a. Draw $\overline{AE}$ on your paper.

b. Draw a circle with center at A and $\overline{AE}$ as a radius. Call the circle C.

c. Is a radius of a circle part of the circle? (No) Why? (The only point of the radius $\overline{AE}$ which is also a point of the circle is point E.)

d. Draw a diameter of your circle. Is a diameter of a circle part of the circle? (The endpoints only are points of the circle.)

5. a. Draw a circle with center marked A and a radius $\overline{AE}$.

b. Can you imagine another circle with center E and radius $\overline{AE}$? (Yes) Is there more than one such circle? (No, since A and E are particular points.)

6. Imagine two circles.

a. Do they have to be in the same plane? (No)

b. Could their intersection be the empty set? (Yes)

c. Could they intersect in exactly one point? (Yes) Illustrate.

d. Could they intersect in exactly two points? (Yes) Illustrate.

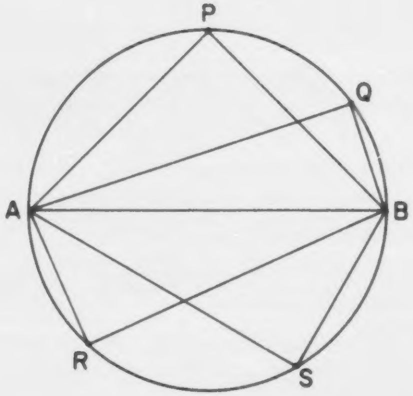
e. Could they intersect in exactly three points? (No) Illustrate.

f. Could they intersect in more than three points? (Yes) Illustrate. (Two circles with the same center and equal radii.)

INSCRIBED ANGLES AND CENTRAL ANGLES

The next objective is to lead the students to recognize that an angle which has its vertex on a circle and which has two other points as endpoints of a diameter of the circle is a 90° angle.

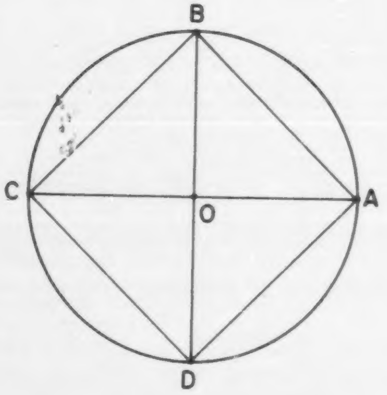
Have the pupils draw a circle and then draw any one of its diameters. Call the ends of the diameter A and B. Then let P, Q, R, and S mark four other points of the circle. Have them draw segments from P, Q, R, and S to both A and B. Then have the students measure very carefully with their protractors the angles APB, AQB, ARB and ASB. Make a record of their measurements on the chalkboard so all the pupils can see the measurements. Did all the pupils get the same measure for each angle? Should they get the same measure? (Yes.) If they could measure accurately, they should get 90 for the measure, in degrees, of each angle.



Does $\angle$ APB have its vertex on the circle? (Yes) Are there two other points of $\angle$ APB which are endpoints of a diameter of the circle? (Yes) Now ask the same questions concerning $\angle$ AQB, $\angle$ ARB, $\angle$ ASB. Now if an angle has its vertex on a circle and two other points which are endpoints of a diameter, what can we say about the size of the angle? (We can say that the angle is a right angle.)

In the drawing below, the polygon ABCD has all its vertices on the circle. $\overline{AC}$ and $\overline{BD}$ are diameters of the circle and $\angle$ BOA is a right angle.

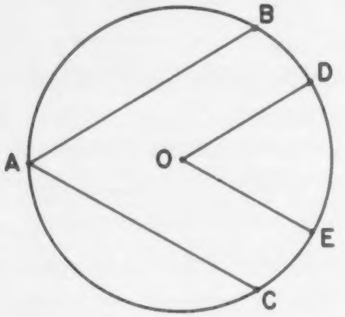
(Have the pupils make such a figure by drawing a circle and any one of its diameters. Then have them draw, by use of the protractor, a second diameter which makes a right angle with the first diameter.) Measure with the protractor each of these inscribed angles: $\angle$ ABC, $\angle$ BCD, $\angle$ CDA, $\angle$ DAB. Do they all seem to be right angles? (Yes) Do $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ appear to be equal? (Yes) If you draw two diameters of a circle which are perpendicular to each other and draw line segments connecting the endpoints of these diameters in the order shown in the drawing, do you think the polygon so formed will always be a square? (Yes)



Now, study with the pupils the pupils' material on INSCRIBED ANGLES AND CENTRAL ANGLES and then proceed to Exercise Set 5.

INSCRIBED ANGLES AND CENTRAL ANGLES

Draw two chords of a circle with point A as one endpoint of both. See the drawing below.



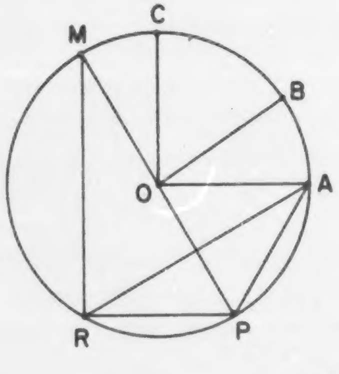
Call the chords AB and AC. The ∠ BAC is called an inscribed angle.

Angle DOE is called a central angle since its vertex is at the center of the circle.

Write the names of the central angles and the names of the inscribed angles that you see in the drawing below. A diameter of the circle is $\overline{MP}$. Make two columns on your paper as suggested below the drawing.

Central Angles

∠ COB
∠ COA
∠ BOA
∠ MOC
∠ MOB
∠ MOA
∠ COP
∠ BOP
∠ AOP



Inscribed Angles

∠ MPR
∠ RMP
∠ ARP
∠ APR
∠ APM
∠ MRP
∠ MRA
∠ RAP

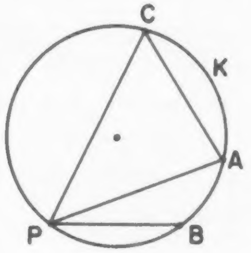
Central Angles

Name at least 7 of these.

Inscribed Angles

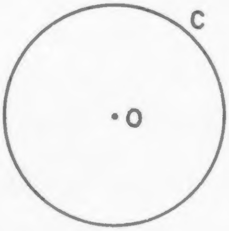
Name at least 8 of these.

If we have an inscribed angle in a circle, we can name the arc in its interior with two letters. In the table below the next drawing we have written the names of some of the inscribed angles and the arc which lies in the interior of the angle.



Inscribed Angle	Arc in its interior
∠ CPA	$\overset{\frown}{CA}$
∠ CPB	$\overset{\frown}{CB}$
∠ ACP	$\overset{\frown}{AP}$

Can you tell how to locate two points of the circle C in the drawing below so that the two arcs into which they divide the circle will be congruent? (Yes. Draw a chord through the center of the circle. A diameter will locate two points so the two arcs will be congruent.)



After you think you have located two such points, ask your teacher if you have done it correctly.

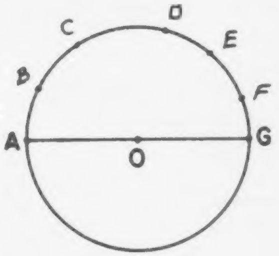
Exercise Set 5

1. a. Draw a circle with center marked O.

   b. Draw a central angle of the circle. Is this central angle a part of the circle? (No. A central angle has its vertex at the center of a circle. Only points A and B are points of the circle.)

   c. Draw an inscribed angle of the circle. Is it a part of the circle? (No. With an inscribed angle; there are three points in common with the circle. They are the points R, S, and T.)

2. Draw a circle as shown below.



   a. Mark the center O and diameter $\overline{AG}$.

   b. Mark a point D on your circle.

   c. On $\widehat{ADG}$ mark points B, C, E, and F.

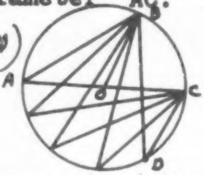
   d. Name at least 3 arcs of the circle. ($\widehat{ABC}$, $\widehat{ACD}$, $\widehat{CDE}$, $\widehat{FGA}$, etc.)

   e. Mark and measure (with a protractor) ∠ ABG, ∠ ACG, ∠ ADG, ∠ AEG, and ∠ AFG.

   f. Do you notice anything surprising about the measures, in degrees, of these inscribed angles? (yes) Explain. (Their measure seems to be 90.)

   g. Make a guess about the measures, in degrees, of all inscribed angles with one ray through one end of a diameter and the other ray through the other end. (Their measure will always be 90.)

3. a. Draw a circle with center marked O and diameter $\overline{AC}$.



b. Mark a point B on the circle. (Answers will vary.)

c. Draw ∠ BOC. What kind of angle is this? (Central angle)

d. By using a protractor, approximate the measure of ∠ BOC. (Answers will vary.)

e. Draw ∠ BAC. What kind of angle is this? (inscribed angle)

f. By using a protractor, approximate the measure of ∠ BAC. (answers will vary.)

g. Mark a point D of $\widehat{ACB}$ which is not a point of $\widehat{ABC}$. Draw inscribed ∠ BDC.

h. By using a protractor, approximate the measure of ∠ BDC. (It should have the same measure as ∠ BAC.)

i. Draw three more inscribed angles with vertices on $\widehat{ADC}$ each having one ray through B and the other one through C. Do you notice anything about the measures of these inscribed angles with common $\widehat{BC}$? (The measures of these inscribed angles with common $\widehat{BC}$ are the same.)

j. Make a guess about measures of all inscribed angles of a given circle with the same $\widehat{BC}$. (They all have the same measure.)

k. Make a guess about the measure of a central angle with $\widehat{BC}$ and the measure of an inscribed angle with $\widehat{BC}$. (The central angle with $\widehat{BC}$ has a measure two times as large as the inscribed angle with $\widehat{BC}$.)

4. Draw a picture of a circle on a piece of paper. Crumple this sheet into a wad. Is the drawing still a picture of a circle? (No) Why? (Because, now all of the points are not in the same plane.)

5. **BRAINTWISTER**

Imagine all the diameters of a circle.

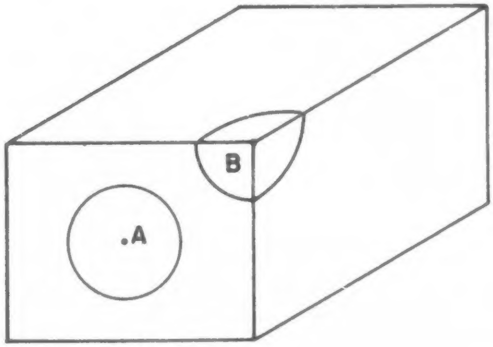
a. What would be their intersection? (the center of the circle)

b. What would be their union? (the circular region)

6. **BRAINTWISTER**

a. Place the sharp point of a compass at a point of one face of a block. Name this point A. With A as a center and a suitable setting on the compass, draw a simple closed curve on this face. Is the result a picture of a circle? (Yes)

b. With this same setting of the compass, draw a simple closed curve on the box using the vertex B as the center. Are all the points of the resulting curve points of a circle? (No) Why? (Because they are not in the same plane.)



STRAIGHTEDGE AND COMPASS CONSTRUCTIONS

CONSTRUCTING A RAY PERPENDICULAR TO A LINE

This section might be begun with the pupils by having them construct a right angle at a point on the line by use of the protractor. This has been done in the study of the Circles, but it may be well to do it again, partly to stimulate the pupil's curiosity concerning the construction of the right angle which they are soon to do without the use of the protractor. The word *construct* as used in this sense may be new to the pupils. They may prefer to use the word *draw*, but lead them to the use of the word construct. The process of constructing is more than drawing. It is the production of a model by the use of instruments, namely straightedge and compass, as we shall think of constructing here. It is important that the pupils make each construction step-by-step in response to the instructions. It is possible the teacher may need to make the construction on the board as he and the pupils read and discuss the instructions together. The pupils may need to be encouraged to make each construction a second time if the first one is not satisfactory. When a satisfactory construction has been produced by the pupil it should be retained by the pupil for use in the exercises that pertain to it.

It is possible that a word of caution concerning the word *perpendicular* may be worthwhile. Certainly this caution should be given to the pupils. In its original usage, the word perpendicular meant a "plumb line direction" or "vertical direction." Consequently, the word perpendicular may still retain, to some extent, the erroneous and narrow meaning of vertical line when instead it should be used in the following proper sense:

Two lines are perpendicular to each other if and only if one of the angles with its vertex at the intersection of the two lines is a right angle. Subsequent considerations will show that the four angles at the intersections of the two lines are right angles and the pupils may make this observation at this time.

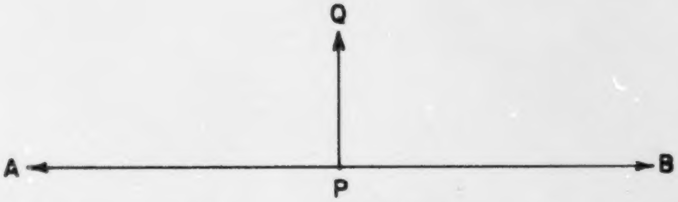
There is no attempt made at this stage of the pupils' development to exhibit a formal proof of the correctness of the results that are stated following any of the constructions. Rigorous proofs are neither desirable nor possible at this time. In their later work in geometry the pupils will recognize that they were led here to make conclusions intuitively and by inference as a result of measurements and that these methods of reaching conclusions will be replaced at the proper time by logical reasoning and deductive argument. The actual steps in all the constructions are very nearly, perhaps precisely, carried out in the same way here that the pupils will perform them in their later study of geometry.

Now proceed to the pupils' section on the construction of a ray perpendicular to a line.

STRAIGHTEDGE AND COMPASS CONSTRUCTIONS
CONSTRUCTING RIGHT ANGLES USING PROTRACTOR

Let us use the protractor to make a right angle with its vertex at the point P on the line AB.

Make a drawing of $\overleftrightarrow{AB}$ on your paper and follow the directions suggested below



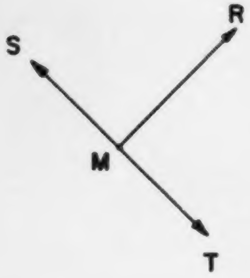
Place the V-point of the protractor at P and the zero ray of the protractor along $\overrightarrow{PB}$. Mark a point, call it Q, on the paper at the 90° mark on either scale of the protractor. Draw $\overrightarrow{PQ}$. Measure $\angle$ BPQ and $\angle$ APQ with the protractor. Is $m \angle BPQ = m \angle APQ = 90$? It should be.

In the drawing that you have made, you may say that $\overrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{AB}$ at P. We write either $\overrightarrow{PQ}$ is $\perp$ to $\overleftrightarrow{AB}$ at P, or $\overleftrightarrow{AB}$ is $\perp$ to $\overrightarrow{PQ}$ at P. Notice the symbol $\perp$ means perpendicular. Saying $\overrightarrow{PQ}$ is $\perp$ to $\overleftrightarrow{AB}$ at P means that $\angle$s APQ and BPQ are right angles. Also saying $\angle$s QPB and QPA are right angles means $\overrightarrow{PQ}$ is $\perp$ to $\overleftrightarrow{AB}$ at P.

Exercise Set 6

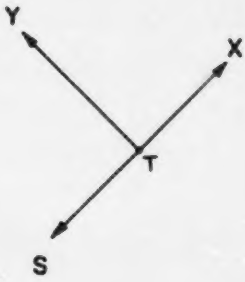
1. Write statements for each drawing as written in a. The angles shown in the drawings are right angles.

a.



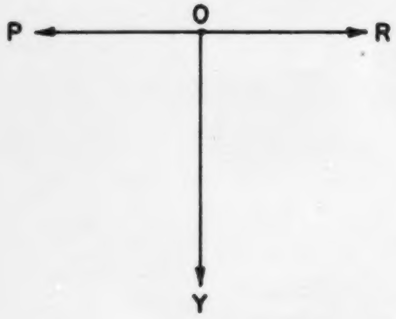
∠ SMR and ∠ RMT are right angles. $\overrightarrow{MR}$ is ⊥ to $\overleftrightarrow{ST}$ at M.

b.



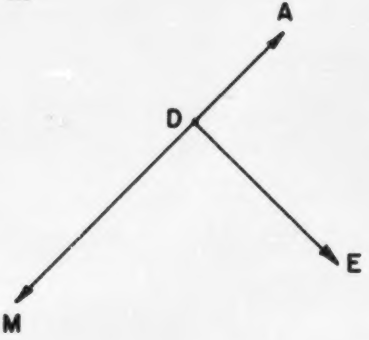
(∠STY and ∠YTX are right angles. $\overline{TY}$ is ⊥ to $\overleftrightarrow{SX}$ at T.)

c.



(∠POY and ∠YOR are right angles. $\overrightarrow{OY}$ is ⊥ to $\overleftrightarrow{PR}$ at O.)

d.



(∠MDE and ∠EDA are right angles. $\overrightarrow{DE}$ is ⊥ to $\overleftrightarrow{MA}$ at D.)

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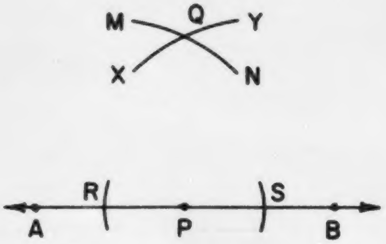
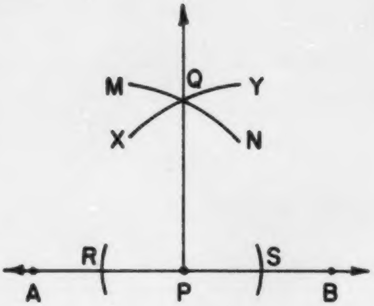
CONSTRUCTING A RAY PERPENDICULAR TO A LINE WITHOUT USING PROTRACTOR

Next let us see how we can draw a ray perpendicular to a line without using the protractor. We shall use a compass and a straightedge.

As you read the directions here, carry out the constructions on your paper. The marks in the drawing will help you to follow instructions.

Choose any point, P, on line AB.





Set the compass point at P and draw two small arcs which will cut $\overleftrightarrow{AB}$ at points R and S. Is $PR \cong PS$? (yes)

Set the compass point at R. Adjust the compass so that the compass setting is greater than the length of $\overline{RP}$. Then draw an arc such as the arc MN in the drawing.

Keep the same compass setting. Set the compass point at S and draw another arc XY in the drawing. The arcs intersect in a point. Call it Q.

Draw $\overrightarrow{PQ}$. The ray $\overrightarrow{PQ}$ is perpendicular to $\overleftrightarrow{AB}$ at P. Your completed drawing should look something like the one above at the right.

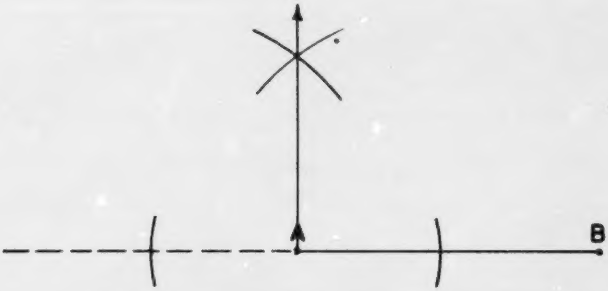
Do you think $\angle SPQ$ and $\angle RPQ$ are right angles? Measure each of them with the protractor. The measure in degrees of each angle should be 90.

This is equivalent to saying that $\overrightarrow{PQ}$ is $\perp$ to $\overleftrightarrow{AB}$ at P from the meaning of perpendicular.

Ray PQ is $\perp$ to $\overleftrightarrow{AB}$. The $\overline{PQ}$ is part of $\overrightarrow{PQ}$, and $\overline{RS}$ is part of $\overleftrightarrow{AB}$. So we can say $\overline{PQ}$ is $\perp$ to $\overline{RS}$ at point P.

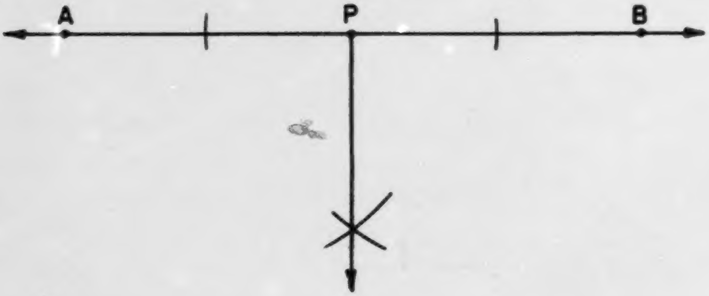
Suppose you wish to construct a line segment $\perp$ to $\overline{AB}$ at a point on $\overline{AB}$. Can you make the construction in the same way that you made the $\overrightarrow{PQ}$ $\perp$ to $\overleftrightarrow{AB}$? (yes)

Next, suppose you wish to construct a line segment $\perp$ to $\overline{AB}$ at point A. Does the drawing below show you how to begin? The dotted segment to the left of A suggests that it was not a part of $\overline{AB}$ but that you needed it in order to make the construction. Now complete the construction on your paper.

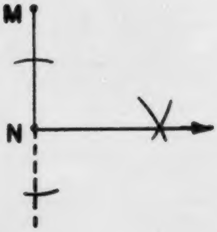


Exercise Set 7

1. Copy $\overleftrightarrow{AB}$ on your paper and construct a ray $\perp$ to $\overleftrightarrow{AB}$ at P. Make your construction so that the ray will be in the half plane below $\overleftrightarrow{AB}$.



2. Copy the segment MN on your paper and construct a segment $\perp$ to $\overline{MN}$ at point N. Make your construction so that the segment will be to the right of the point N.

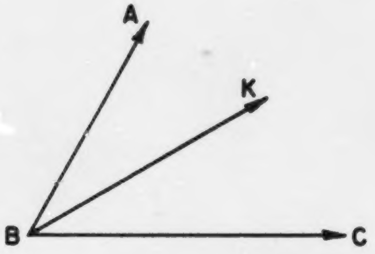


BISECTING AN ANGLE

The detailed instructions for the construction of a ray that bisects a given angle are given in the pupils' text. As has been stated previously, part of the purpose of the constructions in the unit is to provide an experience for the pupil of reading understandingly a list of instructions and following them step-by-step. It is recommended that the teacher and the pupils study the instructions together, with the teacher making the constructions at the board as the pupils make the constructions on their paper. It is possible, of course, that the teacher may find it unnecessary to carry out the constructions at the board. The pupils may be able to read and follow the instructions without much assistance from the teacher and the teacher is the one to be the judge of exactly how this is to be handled. There will undoubtedly be some frustration on the part of some of the pupils in finding that the two angles obtained by their construction do not have equal measures as shown by their protractor. If the difference in the sizes of the two angles is very pronounced, the pupils will need to make a second or even third attempt.

BISECTING AN ANGLE

The ∠ ABC in the drawing below has the measure in degrees of 60. This angle can be made by using the protractor or by making an equilateral triangle. Remember we know the measure in degrees of any one of the angles of any equilateral triangle is 60.



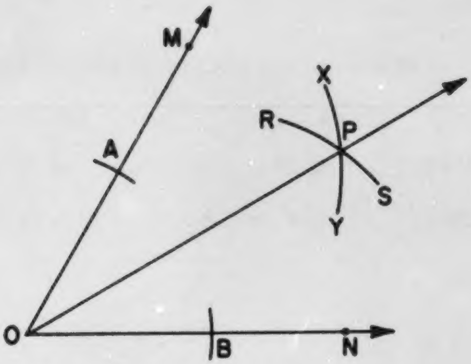
Copy the angle on your paper.

Draw $\overrightarrow{BK}$ so that the measure in degrees of ∠ CBK is 30. Then the measure in degrees of ∠ ABK is also 30.

We say that $\overrightarrow{BK}$ bisects ∠ ABC. The measures of ∠ CBK and ∠ ABK are the same. The sum of their measures in degrees is the measure in degrees of ∠ ABC.

If you have an angle drawn on your paper, could you always use the protractor to help you find a ray which bisects the angle? You could do this if you read the scale on the protractor very accurately.

Now we wish to show how to draw a ray without using the protractor so that the ray will bisect the angle. As you read the directions on the following page, carry out the constructions on your paper. The marks in the next drawing will help you to follow the instructions.



The angle that we are going to bisect is $\angle$ MON. Here are the steps in the construction of the ray that is the bisector.

With the point of the compass set on point O, draw two arcs so that one of them cuts $\overrightarrow{OM}$ at a point (call it A) and the other arc cuts $\overrightarrow{ON}$ at a point (call it B). Keep the setting of the compass unchanged while drawing the two arcs.

With the point of the compass set on point A, and the setting of the compass so that it is greater than one-half the distance from A to B, draw an arc such as $\widehat{XY}$.

With the point of the compass at B, and the setting of the compass the same as used in drawing the arc XY, draw another arc such as $\widehat{RS}$.

The two arcs you have just drawn intersect at a point P. Draw $\overrightarrow{OP}$.

The $\overrightarrow{OP}$ bisects the $\angle$ MON.

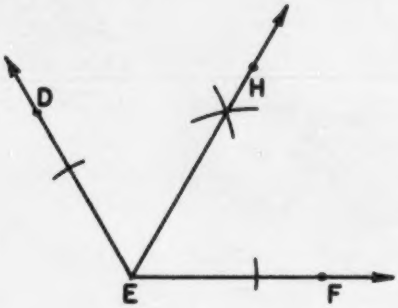
Measure $\angle$ POB and $\angle$ POA with your protractor.

Is m $\angle$ POB in degrees equal m $\angle$ AOP in degrees? (yes)

Is m $\angle$ POB + m $\angle$ AOP = m $\angle$ AOB? (yes)

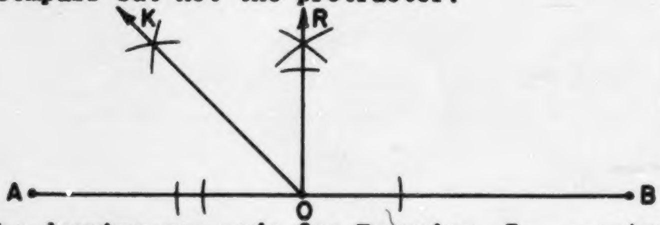
Exercise Set 8

1. Copy the angle DEF on your paper. Then construct $\overrightarrow{EH}$ so that $\overrightarrow{EH}$ will bisect the angle. Use your compass and straightedge but not your protractor.



2. With your protractor measure $\angle DEF$ in the drawing on your paper. Then $\angle HEF$ and $\angle HED$. Is the sum of the measures in degrees of $\angle HEF$ and $\angle HED$ equal to the measure of $\angle DEF$? (yes)

3. On your paper draw $\overline{AB}$ about 4 inches in length. Name a point O on $\overline{AB}$ between A and B. Construct $\overrightarrow{OR}$ $\perp$ to $\overline{OA}$ at O. Use the straightedge and compass but not the protractor.



4. In the drawing you made for Exercise 3, construct a ray OK so it will bisect $\angle AOR$. Use the straightedge and compass but not the protractor.

5. Find by using the protractor the measure of $\angle AOR$ that you constructed in Exercise 3. (90) Then use the protractor to find the measure of $\angle AOK$. (45) Is the measure of $\angle AOR$ twice as large as the measure of $\angle AOK$? (yes)

INSCRIBING A REGULAR HEXAGON IN A CIRCLE

The meaning of the three words inscribing, regular, and hexagon should be made clear to the pupils in the beginning of the construction. However, they are not introduced in the pupils' material until after the construction is completed. Present them in this order: (Prior to the explanation, construct a regular inscribed hexagon in a circle on the board to assist in explaining the meaning of the words)

A hexagon is a polygon of six sides.

A regular hexagon is a hexagon whose sides are congruent segments and whose angles are congruent angles.

Inscribing a regular hexagon in a circle means constructing a regular hexagon so that its sides will be chords of a circle and its angles will be inscribed angles of the circle.

Begin with a circle. The circle is given, and a regular hexagon is to be inscribed in it. Note that we are not beginning with a given hexagon and putting it in a circle.

The circle with which you begin may be a circle with any convenient length for the radius. For the pupils who are working on their paper, a radius of approximately 2 inches will likely be convenient. For the teacher's construction at the board, a circle with a radius of approximately 18 inches will likely be convenient.

The exercises in Exercise Set 9 will help the pupils summarize the important facts to be obtained from the completed construction. Now proceed to the section in the pupils' book on Inscribing a Regular Hexagon in a Circle.

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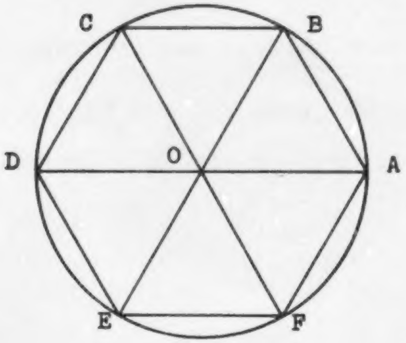
INSCRIBING A REGULAR HEXAGON IN A CIRCLE

Look very carefully at the drawing below. We want you to see some of the many interesting things about the drawing.



Things to be seen in the drawing:

1. A circle with center at point O.
2. Segments OA, OB, OC, OD, OE, and OF are radii of the circle.
3. There are 6 triangles shown with O as a common vertex.
4. Two sides of each of the 6 triangles are radii.
5. One side of each of the 6 triangles is a chord of the circle.
6. The vertices of the polygon ABCDEF are points of the circle.
7. The chords AB, BC, CD, DE, EF, and FA seem to be congruent.
8. All of these 6 chords seem to be of the same length as the radii.
9. The 6 triangles seem to be equilateral triangles.
10. The 6 triangles seem to be congruent to each other.
11. The segments $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ seem to be diameters.
12. The measure in degrees of each of the angles ABC, BCD, CDE, DEF, EFA, FAB seems to be 120.

After you have looked carefully, did you see all of the things that are listed following the drawing? If you saw all twelve of the things listed above and below the drawing, you did very well. But you may have seen more than these. If you did, tell your teacher. Your teacher may have seen some more, too.

As you looked at the drawing, you may have discovered how the drawing can be constructed with the use of compass and straightedge.

The following directions describe how the drawing can be made with compass and straightedge. Follow them carefully and make the construction on your paper. Then save it so that you can use it later.

First draw the circle with center O. Make its radius the length of $\overline{OA}$. Mark a point of your circle and call it A.

Put compass point at A. Use the same compass setting as used in drawing the circle. Make an arc that cuts the circle at point B.

Next put compass point at B, keeping the same compass setting. Make an arc that cuts the circle at C.

Now you see how to mark the point that is named D. Mark it.

Before you mark any of the other points of the circle, do the following:

Draw $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$, $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$.

Are the triangles OAB, ABC, and OCD equilateral triangles? (Yes.) You know that they are because the three sides of each triangle are congruent to each other. Also, the length of each side of the 3 triangles is the same as the length of the radius of the circle.

Measure each angle of one of the triangles with your protractor. The measure, in degrees, of each angle should be 60.

Is $m \angle AOB + m \angle BOC + m \angle COD = 180$? (It should be.)

Place your straightedge along the $\overline{OA}$. Are $\overline{OA}$ and $\overline{OD}$ parts of the same straight line? They should be. Segment AOD is a diameter of the circle. In this unit we will read a line segment that is a diameter with three letters. The letter in the middle names the point at the center of the circle.

Now continue from D to mark points E and F just as you marked points B and C when you started from A.

Draw $\overline{OE}$, $\overline{OF}$, $\overline{DE}$, $\overline{EF}$.

Measure $\angle DOE$ and $\angle EOF$ with your protractor. Is the $m \angle DOE + m \angle EOF = 120$? It should be.

Is the $m \angle FOA = 60$? It should be. With the compass setting the same as the length of the radius of the circle and the point of the compass at F, make an arc. This arc should intersect the circle at A.

One more equilateral triangle each of whose sides is congruent to a radius of the circle can be "fitted into" the circle. One vertex will be at O, one at F, and the third one at A. This completes the construction.

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Now you can see that all the "seem to be" statements are true. These statements are numbered 7, 8, 9, 10, 11, 12.

The polygon ABCDEF is a hexagon. It is called a hexagon because it has six sides. In this hexagon the sides are congruent to each other, and the angles are congruent to each other.

A polygon which has congruent sides and congruent angles is a regular polygon. The polygon you have drawn is a regular hexagon.

Can we say that the polygon ABCDEF is a regular hexagon inscribed in the circle that has its center at O and that the length of each side of the hexagon is the same as the length of the radius of the circle? (yes)

Exercise Set 9

(You will need to use the drawing of the inscribed hexagon that you have constructed or the one drawn in this book.)

1. If the length of the radius of the circle is 1 inch, what is the measure in inches of the perimeter of the hexagon? (6)

2. What is the measure in degrees of each of the angles AOB, BOC, COD, DOE, EOF, and FOA? (60)

3. Draw $\overline{AC}$, $\overline{CE}$, and $\overline{EA}$. Use your compass to compare the lengths of these segments. Are $\overline{AC}$, $\overline{CE}$, and $\overline{EA}$ congruent? (yes) What kind of triangle do we call △ ACE? (equilateral triangle)

4. Construct a ray (call it $\overrightarrow{OP}$) which bisects ∠ AOB. Mark the point Q in which $\overrightarrow{OP}$ intersects the circle. Draw $\overline{AQ}$ and $\overline{BQ}$. Use your compass to compare their lengths. Is $\overline{AQ} \cong \overline{BQ}$? (yes)

5. Does exercise 4 suggest a way that you could use to construct a polygon with 12 congruent sides that would be inscribed in the circle? (yes) Describe the way you would do it. Do not make the construction. (1. You could bisect each of the six central angles and using the vertices of the hexagon, join the 12 points located on the circle.
or 2. Bisect angle AOB and locate point Q. Then using the length of the radius and starting at Q, find 6 more points on the circle.)

6. Does exercise 5 suggest a way that you could use to construct a polygon with 24 congruent sides that would be inscribed in the circle? (yes) Describe the way you would do it. Do not make the construction. (Use methods similar to those explained in exercise 5.)

INSCRIBING A SQUARE IN A CIRCLE

In inscribing a square in a circle, the pupils will be making use of the construction of a perpendicular to a line at a point on the line. Hence, part of the objective is an application of this construction. Also, we wish to emphasize that an angle inscribed in a semi-circle is a right angle. In order to strengthen the idea that a perpendicular is not necessarily a vertical line, the first diameter of the square is chosen so that it is neither vertical or horizontal.

Study with the pupils their instructions on inscribing a square within a given circle. Emphasize that we decide upon some circle in advance and use any convenient length for the radius of the circle. Then follow the instructions step-by-step with the pupils. You may wish to make the different steps in the construction at the board as the pupils make the corresponding steps on their paper. The precision of their constructions are judged by measuring the sides of the constructed square with compass and comparing the lengths. They should be the same. The central angles of the square should be right angles. The measures of the angles are made with the protractor.

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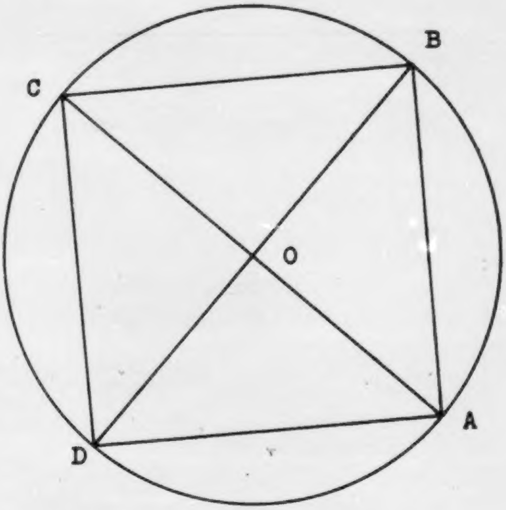
INSCRIBING A SQUARE IN A CIRCLE

Look carefully at the drawing but do not do any measuring now. Make a list of some of the things you see. After you have studied the drawing, then see whether you saw all the things that are listed below the drawing.



Things to be seen in the drawing.

1. A circle with its center at point O.
2. $\overline{AOC}$ and $\overline{BOD}$ are diameters of the circle.
3. $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$ are radii of the circle.
4. $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are chords of the circle.
5. Four triangles OAB, OBC, OCD, and ODA with a common vertex at O.
6. Four triangles ABC, BCD, CDA, and DAB.
7. The angles AOB, BOC, COD, and DOA seem to be right angles.
8. The angles ABC, BCD, CDA, and DAB seem to be right angles.
9. The chords $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ seem to be congruent.

As you looked at the drawing, you may have discovered how it can be constructed with compass and straightedge.

The following directions describe how the construction can be carried out. Follow them carefully and make the construction on your paper. Then save it so you can use it later.

Draw a circle on your paper which will have its radius the same length as the circle in the drawing. Call the center of your circle O and mark a point A on your circle that corresponds to the point A in the drawing.

Draw the diameter that has one of its ends at A. Name the other end of the diameter C. Must the center O be a point of the diameter? (yes)

Construct another diameter which will be $\perp$ to $\overline{AC}$ at O. Use your compass and straightedge but not your protractor. (Remember that you know how to construct a line $\perp$ to another line at a point on it.) Letter the ends of this diameter as in the drawing.

Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. This completes the construction.

Exercise Set 10

1. Use your compass to compare $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Are they congruent? They should be.

2. Use your protractor to measure $\angle$ AOB, $\angle$ BOC, $\angle$ COD, and $\angle$DOA. Is 90 the measure in degrees of each of them? (yes) Is each one of them a right angle? (yes)

3. Use your protractor to measure the $\angle$ ABC, $\angle$ BCD, $\angle$ CDA, and $\angle$ DAB.
Is 90 the measure in degrees of each one of them? (yes)
Is each one of them a right angle? (yes)
Is each one of them inscribed in a semi-circle? (yes) (Remember that an angle is inscribed in a semi-circle if its vertex is a point of the circle and the endpoints of a diameter are points of its rays.)

4. Name eight right triangles in the drawing.
(Triangles AOB, BOC, COD, DOA, ABC, BCD, CDA, DAB)

5. Name eight isosceles triangles in the drawing. (Same as those listed in exercise 4.)

6. Are the triangles you named in exercise 4 the same as you named in Exercise 5? (Yes)

7. Do you see any equilateral triangles in the drawing? (No)

8. Can we say that this quadrilateral ABCD is a regular inscribed polygon? (yes) Is it a square? (yes)

9. With compass and straightedge construct the bisector of $\angle$ AOB. Let P be the name of the point where the bisector intersects the circle.

10. Draw the ray opposite to $\overrightarrow{OP}$ until it intersects the circle at a point between C and D. Name the point Q.

11. Is 45 the measure in degrees of each of the angles AOP, BOP, COQ, and DOQ? (yes)

12. Draw $\overline{AP}$, $\overline{BP}$, $\overline{CQ}$, and $\overline{DQ}$. Use your compass to compare their lengths. Are they congruent segments? (yes)

13. Put the compass point at B. Use the same setting of your compass that you used in Exercise 12 when you compared the length of the 4 segments. Make an arc which will intersect the circle between B and C. Name this point R. Draw $\overline{BR}$ and $\overline{CR}$.

14. Draw the diameter which has one end at point R. Let S be the name of the point at its other end. Draw $\overline{DS}$ and $\overline{SA}$.

15. Is the polygon APBRCQDS an inscribed polygon? (yes) It has eight sides. It is called an octagon. All of its sides are congruent line segments.

16. Use your protractor to measure at least three angles of the octagon. For example, you might measure angles APB, PBR, and BRC. Is 135 the measure in degrees of each one of them? (yes)

17. Can you tell how you could inscribe a polygon of 16 sides in the circle so that all of its sides would be congruent segments? (yes) Describe how it can be done but do not make the drawing. (Follow extended procedure used to draw the octagon.)

18. You have seen how you could inscribe polygons of 4, 8, and 16 sides in the circle. Can you describe how you could inscribe a polygon of more than 16 sides in the circle so that all of the sides of the polygon would be congruent line segments? (yes)

19. Do you think the perimeter of the octagon is greater than the perimeter of the square? (yes) of the 16-sided polygon? (No)

20. Do you think we could inscribe in a circle a polygon with a very, very large number of congruent sides so that its perimeter would be nearly the same as the perimeter of the circle? (yes) Exactly the same? (No)

Chapter 10

REVIEW

PURPOSE OF UNIT

The purpose of this unit is to provide a review of some of the concepts and skills that the pupils have learned in Mathematics in Grade Six. The material provided here is intended to review the study of Numbers and Numerals as indicated by the title of the unit. The review of the geometry studied in Grade Six is fairly well done in Chapter 9. Hence, no review lessons which refer to geometry units are provided here.

TEACHING PROCEDURES

The teaching procedure will depend upon the amount of review that is needed. Some pupils may be able to answer all the questions and work all the problems with little or no assistance from the teacher. Other pupils may have some difficulty. In general, it is suggested that the pupils review a particular unit without any other preparation than they had while studying the unit. After the pupils have responded to the review questions in accord with the given directions, the teacher can determine whether there needs to be some instruction to the entire class and to individual pupils. Pupils who have difficulty with the review items should be encouraged to turn to their copy of the chapter which gave them difficulty in order to correct their own errors. If the pupils have done well enough to indicate no review of a particular chapter is needed, they may well undertake the next one.

Chapter 10

REVIEW

WORKING WITH EXPONENTS

Exercise Set 1

Write the numerals 1 through 12 on your paper. Decide if each of the following mathematical sentences is true or false. Then write true or false opposite each numeral on your paper.

1. $3^3 = 3 \times 3$ (false)
2. $4^5 = 4 \times 4 \times 4 \times 4 \times 4$ (true)
3. $3 \times 2^2 = 3 \times 2 \times 3 \times 2$ (false)
4. $5^3 \times 5^2 = 5^6$ (false)
5. $8 \times 2^3 = 8 \times 2 \times 2 \times 2$ (true)
6. $(2 \div 3)^4 = 2^4 \div 3^4$ (true)
7. $2^8 = 8^2$ (false)
8. $2^4 = 4^2$ (true)
9. $6^2 = 2^6$ (false)
10. $8^2 - 3^2 = (8 - 3)^2$ (false)
11. $8 \times 3^2 = (8 \times 3)^2$ (false)
12. $4 \times 3^2 = 4 \times 4 \times 3 \times 3$ (false)

In Exercise 13 through 24 write each expression as a decimal numeral.

13. $3^2 \div 1$ (9)
14. $11 - 3^2$ (2)
15. $2^4 - 4^2$ (0)
16. $8^2 - 6^2$ (28)
17. $8^3 - 6^3$ (296)
18. $(3^4 \div 1) - 7^2$ (32)
19. $5^3 - (4^2 - 3^2)$ (118)
20. $(3 \div 4)^2 \div 5$ ($\frac{9}{80}$ or 0.1125)
21. $88 - (3 - 3)^8$ (88)
22. $(18 - 17)^{10} \div (2 \div 1)^4$ ($\frac{1}{16}$ or 0.0625)
23. $2^2 \div (4^2 \div 2^2)$ (1)
24. $4^2 - (3^2 - 2^2)$ (11)

Write >, or <, or = for each blank in Exercise 25 through 38 so that each mathematical sentence will be true.

25. $3^2 + 1 \underline{\quad = \quad} 1 + 3^2$

26. $18^5 + 21^7 \underline{\quad = \quad} 21^7 + 18^5$

27. $2^{10} + (2^{15} + 2^{21}) \underline{\quad = \quad} (2^{10} + 2^{15}) + 2^{21}$

28. $9^{25} + 0 \underline{\quad = \quad} 0 + 9^{25}$

29. $64^{20} \underline{\quad = \quad} 64^{20} \times 1$

30. $10^8 - 1 \underline{\quad < \quad} 10^8 + 1$

31. $15^7 + 8^7 \underline{\quad > \quad} 15^7 - 8^7$

32. $(4 \times 5)^2 \underline{\quad = \quad} 4^2 \times 5^2$

33. $6^2 - 4^2 \underline{\quad = \quad} (6 - 4) \times (6 + 4)$

34. $0^{896} \underline{\quad = \quad} 0^{895}$

35. $7^2 + 5^2 \underline{\quad < \quad} (7 + 5)^2$

36. $1^{502} \underline{\quad < \quad} 502^1$

37. $(1 + 1 + 1)^4 \underline{\quad = \quad} 3^4$

38. $(1 - 1 + 1)^8 \underline{\quad > \quad} 0$

Exercise Set 2

1. What counting number does n stand for if $4^n = 16$?

You could think, "$4^1 = 4$, so n is not 1.

$4^2 = 16$, so $n = 2$."

What counting number does n stand for so that each of these mathematical sentences will be true? Write your work as shown for Exercise a.

a. $4^n = 16$ $\quad$ $4 \times 4 = 16$ $\quad$ $4^2 = 16$ $\quad$ $n = 2$

b. $7^n = 49$ ($n = 2$)

c. $2^n = 8$ ($n = 3$)

d. $2^n = 32$ ($n = 5$)

e. $10^n = 1000$ ($n = 3$)

f. $6^n = 216$ ($n = 3$)

g. $12^n = 144$ ($n = 2$)

h. $4^n = 1024$ ($n = 5$)

i. $3^n = 243$ ($n = 5$)

2. Write each of these expressions as a decimal numeral.

a. n^2, if $n = 6$ (36)

b. 2^n, if $n = 6$ (64)

c. n^3, if $n = 5$ (125)

d. 3^n, if $n = 5$ (243)

e. $3^n + 4$, if $n = 2$ (13)

f. $6^n + n^6$, if $n = 2$ (100)

g. $3^n - 3^n$, if $n = 4$ (0)

h. $100^n - n^{100}$, if $n = 1$ (99)

3. What counting number does n stand for so that $2^n + 1 = 9$?
You could think $2^1 + 1 = 3$, so n is not 1.
$2^2 + 1 = 5$, so n is not 2. $2^3 + 1 = 9$, so $n = 3$.

What counting number does n stand for so that each of the mathematical sentences Exercise a through i will be true? Write your work as shown for Exercise a.

a. $\underline{3^n - 1 = 8}$ $\underline{(3 \times 3) - 1 = 8}$ $\underline{3^2 - 1 = 8}$ $\underline{n = 2}$

b. $2^n - 1 = 3$ $(n=2)$

c. $4^n + 3 = 19$ $(n=2)$

d. $6^n - 4 = 2$ $(n=1)$

e. $2^n + 1 = 33$ $(n=5)$

f. $5^n - 2^3 = 17$ $(n=2)$

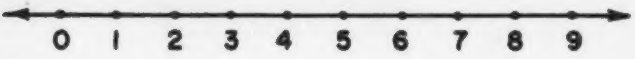
g. $2^n - 2^4 = 16$ $(n=5)$

h. $2^n - 2^5 = 0$ $(n=5)$

i. $3^n + 10 = 37$ $(n=3)$

WORKING WITH WHOLE NUMBERS AND INTEGERS

Exercise Set 3



Above is a graph of a set of whole numbers. Let us call it Set A. Set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. In Exercise 1 and 2 you will be thinking about these whole numbers and only these whole numbers.

Think about the whole numbers in Set A.

Which are described by the mathematical sentence

$$n + 2 = 6?$$

The answer is 4, since $4 + 2 = 6$.

Which are described by $n < 2$?

The answer is 1 and 0, since

$$1 < 2 \quad \text{and} \quad 0 < 2.$$

1. Which members of Set A are described in Exercise a through j?

a. $n > 2$ (3, 4, 5, 6, 7, 8, 9)

b. $n - 2 = 5$ (7)

c. $n < 0$ (none)

d. $n > (5 + 2)$ (3, 4, 5, 6, 7, 8, 9)

e. $2 + n < 4$ (1, 2, 3, 4, 5, 6, 7, 8, 9)

f. $n > 5$ and < 7 (6)

g. $n < 2$ and > 0 (1)

h. $n < 6$ and > 0 (1, 2, 3, 4, 5)

i. $n > 3$ and < 7 (4, 5, 6)

j. $n < 6$ and > 7 (none)
(be careful)

The mathematical sentence $6 < 8 < 11$ means 6 is less than 8 and 8 is less than 11. $6 < 8 < 11$ can be written $6 < 8$ and $8 < 11$.

The mathematical sentence $3 < n < 5$ means to find n so n is greater than 3 and n is less than 5. The only member of Set A for which this is true is 4.

2. Which members of Set A are described in Exercise a through i?

a. $2 < n < 4$ (3)

b. $0 < n < 2$ (1)

c. $5 < n < 7$ (6)

d. $2 < n < 6$ (3, 4, 5)

e. $5 < n < 8$ (6, 7)

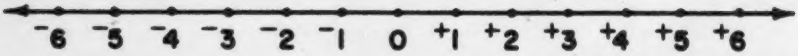
f. $3 < n < 7$ (4, 5, 6)

g. $0 < n < 1$ (none)

h. $6 < n < 7$ (none)

i. $7 < n < 8$ (none)

Exercise Set 4



Above is a number line of a set of integers. The members of the set are the integers that are greater than $^{-}7$ and less than $^{+}7$. Call this Set I.

1. Which of these Sets are shown on the above number line?
(a, b, e)

   a. All integers that are greater than $^{-}1$ and less than $^{+}4$.

   b. All integers that are less than $^{-}2$ and greater than $^{-}5$.

   c. All integers that are greater than $^{-}4$ and greater than $^{+}4$.

   d. All integers that are less than 0.

   e. All integers that are greater than $^{-}5$ and less than $^{-}3$.

2. What integers, if any, of Set I are described by these statements?

   a. All integers less than $^+2$ and greater than 0. $(^+1)$

   b. All integers less than $^-5$ and greater than $^-4$. (None)

   c. All integers greater than 0 and less than 0. (none)

   d. All integers less than $^+6$ and greater than $^+3$. $(^+4, ^+5)$

3. If $^-2 < n < ^+1$, and n is an integer, which integers are greater than $^-2$ and less than $^+1$? The answer is $^-1$ and 0.

4. For each of these, find n if n is a member of the set of integers,

   $\{^-6, ^-5, ^-4, ^-3, ^-2, ^-1, 0, ^+1, ^+2, ^+3, ^+4, ^+5, ^+6\}$

   a. $^+4 > n > ^+2$ $(+3)$

   b. $^-3 < n < ^+2$ $(-2, ^-1, 0, ^+1)$

   c. $^+3 < n < ^+6$ $(^+4, ^+5)$

   d. $^-5 < n < ^-2$ $(-4, ^-3)$

   e. $^-3 < n < 0$ $(^-2, ^-1)$

   f. $^-1 < n < ^+3$ $(0, ^+1, ^+2)$

Exercise Set 5

Answer each of these questions. You may write a mathematical sentence as you answer, if you wish to do so.

1. What integer is 2 greater than the opposite of $^{+}4$? ($^{-}2$)

2. What integer is 2 less than the opposite of $^{-}4$? ($^{+}2$)

3. The integer $^{+}2$ is 2 greater than a certain integer which we will represent by A. Also $^{+}2$ is 2 greater than the opposite of A. What integer does A represent? (0)

4. What integer is 6 greater than the opposite of $^{+}2$? ($^{+}4$)

5. The sum of two integers is $^{+}5$. Both addends are positive integers. Write all possible mathematical sentences.

   ($^{+}5 = {}^{+}1 + {}^{+}4$
   $^{+}5 = {}^{+}2 + {}^{+}3$
   $^{+}5 = {}^{+}3 + {}^{+}2$
   $^{+}5 = {}^{+}4 + {}^{+}1$)

6. The sum of two integers is $^{-}6$. Both addends are negative integers. Write all possible mathematical sentences.

   ($^{-}6 = {}^{-}1 + {}^{-}5$
   $^{-}6 = {}^{-}2 + {}^{-}4$
   $^{-}6 = {}^{-}3 + {}^{-}3$
   $^{-}6 = {}^{-}4 + {}^{-}2$
   $^{-}6 = {}^{-}5 + {}^{-}1$)

7. The sum of two integers is $^{+}8$. Both addends are negative integers. What are the addends? Does this problem have an answer? (No) If your answer is no, state a reason. (The sum of any two negative integers gives a negative integer.)

Exercise Set 6

On a sheet of graph paper, draw the x-axis and the y-axis. Locate the points described below and label each point with its name, but not with its ordered pair.

1. Point E is 5 units above the x-axis and 4 units to the right of the y-axis. [E($^{+}4$, $^{+}5$)]

2. Point R is on the y-axis and 8 units below the point whose coordinates are (0, $^{+}13$). [R(0,$^{+}5$)]

3. The x-coordinate of C is the opposite of $^{+}8$. Its y-coordinate is the opposite of $^{-}5$. [C ($^{-}8$, $^{+}5$)]

4. Point T is the reflection of the point with coordinates ($^{-}12$, $^{+}5$) in the y-axis. [T($^{+}12$, $^{+}5$)]

5. Another point, R, is 6 units above and 9 units to the left of the point with coordinates ($^{+}7$, $^{-}1$). [R($^{-}2$ $^{+}5$)]

6. Another point, C, is on the line segment joining E and T. It is midway between E and T. [C($^{+}8$, $^{+}5$)]

7. Point O has as coordinates integers that are opposites. The second member of this ordered pair is the opposite of $^{-}5$. [O($^{-}5$, $^{+}5$)]

If you have located the points correctly you will find that:

a. All points are on the same straight line.

b. A word is spelled by the letters. Write the word? (CORRECT)

WORKING WITH RATIONAL NUMBERS

Exercise Set 7

The Ancient Egyptians used only unit fractions, for the most part. A unit fraction is a fraction which names a rational number, such as $\frac{1}{2}$, $\frac{1}{9}$, $\frac{1}{15}$, or $\frac{1}{297}$ that has a numerator of 1.

The Egyptians did not have a symbol for rational numbers such as $\frac{3}{4}$. They had to think of $\frac{3}{4}$ as $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, the sum of rational numbers named by unit fractions. They could have thought of $\frac{3}{4}$ as $\frac{1}{2} + \frac{1}{4}$.

In the exercises below you will be asked to name some numbers as the sum of two or more rational numbers whose numerators are 1. For this Exercise Set you are not to use $\frac{1}{1}$ as a unit fraction. You may have to try a few times for each exercise before you get the right answer.

1. Write each of these as the sum of two rational numbers named by unit fractions.

   a. $\frac{2}{5}$ $\left(\frac{1}{5} + \frac{1}{5}\right)$ b. $\frac{2}{8}$ $\left(\frac{1}{8} + \frac{1}{8}\right)$ c. $\frac{1}{2}$ $\left(\frac{1}{4} + \frac{1}{4}\right)$ d. $\frac{1}{3}$ $\left(\frac{1}{6} + \frac{1}{6}\right)$ e. $\frac{10}{20}$ $\left(\frac{1}{4} + \frac{1}{4}\right)$

2. Write each of these as the sum of two rational numbers named by unit fractions in two different ways. An answer for $\frac{3}{8}$ is shown in the box.

   a. $\frac{2}{3}$ $\left(\frac{1}{3} + \frac{1}{3}\ \text{or}\ \frac{1}{2} + \frac{1}{6}\right)$ b. $\frac{5}{12}$ $\left(\frac{1}{3} + \frac{1}{12}\ \text{or}\ \frac{1}{4} + \frac{1}{6}\right)$ c. $\frac{7}{12}$ $\left(\frac{1}{2} + \frac{1}{12}\ \text{or}\ \frac{1}{3} + \frac{1}{4}\right)$

   (These are the correct answers.)

$$\boxed{\begin{aligned}\frac{3}{8} &= \frac{1}{4} + \frac{1}{8}\\ \frac{3}{8} &= \frac{1}{3} + \frac{1}{24}\end{aligned}}$$

3. Copy and fill in the blanks with rational numbers named by unit fractions.

$$\frac{3}{8} = \frac{1}{4} + \frac{1}{8}$$

a. $\frac{3}{8} = \frac{1}{4} + \frac{1}{16} + \underline{\left(\frac{1}{16}\right)}$ c. $\frac{3}{8} = \underline{\left(\frac{1}{8}\right)} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$

b. $\frac{3}{8} = \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \underline{\left(\frac{1}{32}\right)}$

4. Write $\frac{2}{3}$ as the sum of two equal rational numbers named by unit fractions. $\left(\frac{1}{3} + \frac{1}{3}\right)$

Write $\frac{2}{3}$ as the sum of four equal rational numbers named by unit fractions. $\left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right)$

5. Write $\frac{3}{4}$ as the sum of two rational numbers named by unit fractions; $\left(\frac{1}{2} + \frac{1}{4}\right)$ as the sum of three rational numbers named by unit fractions; $\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right)$ as the sum of four rational numbers named by unit fractions. $\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}\right)$

BRAINTWISTERS

6. What is the largest number you can write as the sum of two rational numbers named by unit fractions? $\left(\frac{1}{2} + \frac{1}{2}\right)$

7. What is the largest number you can write as the sum of two rational numbers named by two different unit fractions? $\left(\frac{1}{2} + \frac{1}{3}\right)$

8. What is the largest number you can write as the sum of three rational numbers named by three different unit fractions? $\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$

Exercise Set 8

Copy the product expression shown below.

a. Draw a line under each for which the product is less than the first factor. (1 b,c ; 2a,b ; 3c ; 4b,c)

b. Draw a ring around each for which the product is greater than the second factor. (1b ; 2a ; 3a,bc ; 4b,c)

	(a)	(b)	(c)
1.	$\frac{2}{3} \times 1\frac{7}{8}$ $\left(\frac{5}{4}\right)$	$\underline{3\frac{2}{5} \times \frac{1}{2} \quad \left(\frac{17}{10}\right)}$	$\underline{\frac{3}{4} \times \frac{1}{8} \quad \left(\frac{3}{32}\right)}$
2.	$\underline{12\frac{9}{11} \times \frac{4}{5} \quad \left(\frac{564}{55}\right)}$	$\underline{\frac{1}{8} \times \frac{1}{9} \quad \left(\frac{1}{72}\right)}$	$\frac{4}{9} \times \frac{9}{4}$ (1)
3.	$5\frac{2}{6} \times \frac{11}{4}$ $\left(\frac{44}{3}\right)$	$1\frac{7}{8} \times \frac{9}{8}$ $\left(\frac{135}{64}\right)$	$\underline{\frac{15}{4} \times \frac{3}{8} \quad \left(\frac{45}{32}\right)}$
4.	$\frac{2}{3} \times \frac{9}{2}$ (3)	$\underline{\frac{11}{3} \times \frac{3}{11} \quad (1)}$	$\underline{\frac{15}{8} \times \frac{1}{3} \quad \left(\frac{5}{8}\right)}$

5. Study your answers for the above exercises. Then copy and complete these statements with the words "greater" or "less" in the blanks. Answers are about products of two factors.

   a. If the first factor is less than 1, the product is (less) than the second factor.

   b. If the second factor is less than 1, the product is (less) than the first factor.

   c. If the first factor is greater than 1, the product is (greater) than the second factor.

   d. If the second factor is greater than 1, the product is (greater) than the first factor.

Exercise Set 9

1. What number is n so each mathematical sentence is true?

a. $\frac{1}{2} \times \frac{2}{3} = n$ $\left(\frac{1}{3}\right)$ b. $\frac{7}{4} \times \frac{3}{8} = n$ $\left(\frac{21}{32}\right)$ c. $\frac{5}{6} \times \frac{9}{2} = n$ $\left(\frac{15}{4}\right)$

d. $1\frac{1}{2} \times \frac{5}{3} = n$ $\left(\frac{5}{2}\right)$ e. $3\frac{2}{5} \times 1\frac{3}{10} = n$ $\left(\frac{221}{50}\right)$ f. $2\frac{3}{4} \times 5\frac{1}{3} = n$ $\left(\frac{44}{3}\right)$

g. $\frac{1}{2} \times \frac{3}{2} = n$ $\left(\frac{3}{4}\right)$ h. $\frac{3}{5} \times \frac{7}{4} = n$ $\left(\frac{21}{20}\right)$ i. $\frac{1}{6} \times 2\frac{1}{2} = n$ $\left(\frac{5}{12}\right)$

j. $3\frac{3}{4} \times \frac{5}{2} = n$ $\left(\frac{75}{8}\right)$ k. $\frac{9}{7} \times 1\frac{1}{3} = n$ $\left(\frac{12}{7}\right)$ l. $\frac{11}{4} \times 2\frac{1}{5} = n$ $\left(\frac{121}{20}\right)$

2. Find a fraction name for each of the following.

a. $(4 \times n) + 1$ if $n = \frac{1}{2}$ $\left(\frac{3}{1}\right)$ e. $\left(\frac{5}{4} \times n\right) + \frac{7}{3}$ if $n = \frac{4}{3}$ $\left(\frac{4}{1}\right)$

b. $\left(n \times \frac{2}{3}\right) - 1$ if $n = \frac{3}{2}$ (0) f. $\left(n \times \frac{7}{10}\right) + \frac{8}{20}$ if $n = \frac{1}{2}$ $\left(\frac{3}{4}\right)$

c. $\left(\frac{1}{2} \times n\right) + \frac{1}{2}$ if $n = \frac{1}{3}$ $\left(\frac{2}{3}\right)$ g. $\frac{13}{3} - (n \times 2)$ if $n = \frac{5}{3}$ (1)

d. $\frac{6}{3} - (n \times 2)$ if $n = \frac{2}{3}$ $\left(\frac{2}{3}\right)$ h. $n - \left(\frac{4}{5} \times \frac{2}{3}\right)$ if $n = \frac{14}{15}$ $\left(\frac{2}{5}\right)$

3. Whole-number exponents may be used with fractions as the base. For example,

the meaning of $\left(\frac{3}{4}\right)^2$ and $\left(\frac{2}{5}\right)^3$ is:

$$\left(\frac{3}{4}\right)^2 = \frac{3}{4} \times \frac{3}{4} \qquad \left(\frac{2}{5}\right)^3 = \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5}$$

Write a fraction name for each of the expressions in Exercise a through l.

a. $\left(\frac{1}{2}\right)^2$ $\left(\frac{1}{4}\right)$

b. $\left(\frac{5}{6}\right)^2$ $\left(\frac{25}{36}\right)$

c. $\left(\frac{2}{3}\right)^3$ $\left(\frac{8}{27}\right)$

d. $\left(\frac{7}{8}\right)^2$ $\left(\frac{49}{64}\right)$

e. $\left(\frac{7}{2}\right)^2$ $\left(\frac{49}{4}\right)$

f. $\left(\frac{8}{3}\right)^2 - 1$ $\left(\frac{55}{9}\right)$

g. $\left(\frac{5}{2}\right)^3 + 1$ $\left(\frac{133}{8}\right)$

h. $1 - \left(\frac{1}{2}\right)^4$ $\left(\frac{15}{16}\right)$

i. $\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2$ $\left(\frac{3}{9} \text{ or } \frac{1}{3}\right)$

j. $\left(\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2$ $\left(\frac{20}{25} \text{ or } \frac{4}{5}\right)$

k. $\left(\frac{9}{10}\right)^2 + \left(\frac{11}{10}\right)^2$ $\left(\frac{202}{100} \text{ or } \frac{101}{50}\right)$

l. $\frac{1}{2} + \left(\frac{3}{2}\right)^2$ $\left(\frac{11}{4}\right)$

Exercise Set 10

The mathematical sentence $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$ is a statement about a relation among $\frac{2}{3}$, $\frac{5}{7}$ and $\frac{10}{21}$. The mathematical sentences

$$\frac{10}{21} \div \frac{5}{7} = \frac{2}{3}$$

or

$$\frac{10}{21} \div \frac{2}{3} = \frac{5}{7}$$

are also statements about a relation among the same rational numbers $\frac{2}{3}$, $\frac{5}{7}$ and $\frac{10}{21}$.

Similarly $\frac{1}{2} \times n = \frac{3}{5}$ and $n = \frac{3}{5} \div \frac{1}{2}$ are both statements about a relation among $\frac{3}{5}$, the product, $\frac{1}{2}$, the known factor and n, the unknown factor. One way of finding the unknown factor, n, so $\frac{1}{2} \times n = \frac{3}{5}$ is shown in the box below.

$$\boxed{\begin{aligned} \frac{1}{2} \times n &= \frac{3}{5} \\ n &= \frac{3}{5} \div \frac{1}{2} \\ &= \frac{3}{5} \times \frac{2}{1} = \frac{6}{5} \end{aligned}}$$

1. In Exercise a through f find n so that each mathematical sentence is true.

a. $\frac{3}{4} \times n = 6$ $(n=8)$

b. $n \times \frac{1}{5} = 8$ $(n=40)$

c. $\frac{2}{3} \times n = \frac{4}{7}$ $(n=\frac{6}{7})$

d. $\frac{11}{4} = n \times \frac{3}{4}$ $(n=\frac{11}{3})$

e. $2\frac{1}{2} = \frac{4}{6} \times n$ $(n=\frac{15}{4})$

f. $3\frac{2}{3} = 1\frac{1}{2} \times n$ $(n=\frac{22}{9})$

2. The mathematical sentences in Exercise a through f are true. In which of these mathematical sentences is n a whole number? (a, b, e, f) Try to find an answer without writing them on your paper.

a. $\frac{1}{2} \times n = 4$ $(n=8)$

b. $n \times \frac{2}{3} = \frac{6}{3}$ $(n=3)$

c. $n \times \frac{2}{3} = \frac{3}{3}$ $(n=\frac{3}{2})$

d. $\frac{3}{4} \times n = 2$ $(n=\frac{8}{3})$

e. $\frac{3}{4} \times n = 3$ $(n=4)$

f. $\frac{3}{4} \times n = 6$ $(n=8)$

Exercise Set 11

1. Find the unknown factor for each of exercises a through f. Write your answers in this form: $\frac{2}{3} \div \frac{5}{4} = \frac{8}{15}$. Use whatever other steps you need but do not write them on your paper.

a. $\frac{3}{4} \div \frac{9}{4}$ $(\frac{1}{3})$ c. $\frac{1}{2} \div \frac{9}{2}$ $(\frac{1}{9})$ e. $\frac{11}{9} \div \frac{55}{9}$ $(\frac{1}{5})$

b. $\frac{5}{8} \div \frac{15}{8}$ $(\frac{1}{3})$ d. $\frac{7}{3} \div \frac{35}{3}$ $(\frac{1}{5})$ f. $\frac{9}{6} \div \frac{72}{6}$ $(\frac{1}{8})$

2. In each of the exercises above, is the product or the known factor the greater number? (the known factor) Is the unknown factor less than or greater than 1? (less than 1)

3. Study your answers to Exercise 2. Copy and fill in the blank below so that the statement will be true. In a division example it seems that if the known factor is larger than the product, the unknown factor is (less than 1).

4. Follow the instructions of Exercise 1 to find the unknown factor. See if you can write the unknown factor by studying your answers to Exercise 1.

a. $\frac{9}{4} \div \frac{3}{4}$ (3) d. $\frac{35}{3} \div \frac{7}{3}$ (5)

b. $\frac{15}{8} \div \frac{5}{8}$ (3) e. $\frac{55}{9} \div \frac{11}{9}$ (5)

c. $\frac{9}{2} \div \frac{1}{2}$ (9) f. $\frac{72}{6} \div \frac{9}{6}$ (8)

5. In each of the exercises in problem 4 is the product or the known factor the greater number? (product) Is the unknown factor less than or greater than 1? (greater than 1)

6. Study your answers to Exercise 5. Copy and fill in the blanks so this statement will be true.

In a division example it seems that if the known factor is less than the product, the unknown factor is (greater than 1).

7. Write the unknown factor for each of the exercises below.

a. $\frac{32}{3} \div \frac{1}{3}$ (32)

b. $\frac{32}{3} \div \frac{2}{3}$ (16)

c. $\frac{32}{3} \div \frac{4}{3}$ (8)

d. $\frac{32}{3} \div \frac{8}{3}$ (4)

e. $\frac{32}{3} \div \frac{16}{3}$ (2)

f. $\frac{32}{3} \div \frac{32}{3}$ (1)

Study your answers, then fill in the blanks so each statement will be true.

If the known factor is multiplied by 2 and the product is unchanged, the unknown factor is (divided by 2).

If the known factor is divided by 2 and the product is unchanged, the unknown factor is (multiplied by 2).

Exercise Set 12

Study this set of numbers:

{1, 2, 4, 8, 16, 32, ...}

There is a rule to use in order to find a member of this set. It is: Start with 1; multiply 1 by 2 to get the next member. Any member is 2 times the member before it.

For the set {1, 4, 7, 10, 13, ...} you may think of the rule as, "Start with 1 and add 3."

Write the first 5 members of each of these sets of numbers.

1. Start with 2 and multiply by $2\frac{1}{2}$. $(2, 5, 12\frac{1}{2}, 31\frac{1}{4}, 78\frac{1}{8})$
2. Start with $1\frac{7}{8}$ and divide by $\frac{1}{2}$. $(1\frac{7}{8}, 3\frac{3}{4}, 7\frac{1}{2}, 15, 30)$
3. Start with 288 and divide by $\frac{4}{3}$. $(288, 216, 162, 121\frac{1}{2}, 91\frac{1}{8})$
4. Start with 3.7 and multiply by 6. (3.7, 22.2, 133.2, 799.2, 4795.2)
5. Start with .19208 and multiply by .7.
(0.19208, 0.134456, 0.0941192, 0.06588344, 0.046118408)

6. **BRAINTWISTER**.

Start with 2.1. Multiply by .4 to find the second member of the set. Add 1.2 to this second member of the set. Then repeat. The first three members of the set are: 2.1, .84, 2.04. Write the next five members.
(2.1, 0.84, 2.04, 0.816, 2.016, 0.8064, 2.0064, 0.80256)

Exercise Set 13

Tell whether each statement is always true, sometimes true, or never true.

1. The result of multiplying two rational numbers is a rational number. (always true)

2. The result of multiplying a rational number which is a whole number and a rational number less than 1 is greater than the result of multiplying two rational numbers less than 1. (always true)

3. If the product is doubled and the known factor is doubled, the unknown factor is also doubled. (never true)

4. In a division example the product is larger than the unknown factor. (sometimes true. $12 \div 4 = 3$ and $12 > 3$
$12 \div \frac{1}{4} = 48$ and $12 < 48$)

5. If the known factor is larger than the product, the unknown factor is less than 1. (always true)

6. If one factor is less than 1 and the other is greater than 1, the product is less than 1. (sometimes true)

7. If the result of multiplying two factors is greater than 1, at least one factor is greater than 1. (always true)

8. If the unknown factor is less than 1, both product and known factor are less than 1. (sometimes true)

9. If the known factor is multiplied by 10 and the product divided by 10, the unknown factor is unchanged. (always true)

P618

Exercise Set 14

$$4 \div 1\tfrac{2}{3} = 4 \div \tfrac{5}{3} = \tfrac{1}{4} \times \tfrac{5}{3} = \tfrac{5}{12}$$

WRONG

We all make silly mistakes like the one above. Often estimating answers helps us avoid silly errors. For example, for $4 \div 1\frac{2}{3}$ you could think

$4 \div 1\frac{2}{3}$ is between $4 \div 1$ and $4 \div 2$,

so it is between 4 and 2.

You now see that $\frac{5}{12}$ is not between 4 and 2 so you have made a mistake.

Copy this chart. Fill in the blanks. Exercise 1 is done for you.

	Division	Unknown factor is between	Unknown factor is between	Unknown factor is close to
1.	$8 \div 3\frac{1}{4}$	$8 \div 4$ and $8 \div 3$	2 and 3	2
2.	$12 \div 3\frac{2}{5}$	$12 \div 4$ and $12 \div 3$	3 and 4	3
3.	$23\frac{7}{8} \div 6\frac{3}{4}$	$24 \div 7$ and $24 \div 6$	3 and 4	4
4.	$15\frac{5}{8} \div 1\frac{7}{12}$	$16 \div 2$ and $16 \div 1$	8 and 16	8
5.	$16\frac{5}{6} \div 6\frac{1}{2}$	$17 \div 7$ and $17 \div 6$	2 and 3	3
6.	$46\frac{2}{3} \div 7\frac{1}{6}$	$47 \div 8$ and $47 \div 7$	6 and 7	6
7.	$19\frac{4}{12} \div 4\frac{5}{8}$	$20 \div 5$ and $20 \div 4$	4 and 5	4

Exercise Set 15

In his science class Bob learned that a gallon of water weighs 8.36 pounds. He also found the following information about milk, turpentine, and gasoline:

A gallon of milk weighs 1.03 times as much as a gallon of water.

A gallon of turpentine weighs .87 times as much as a gallon of water.

A gallon of gasoline weighs .67 times as much as a gallon of water.

Bob weighed a 10-gallon tank that he had. It weighed 18.7 pounds. A 5-gallon can weighed 10.4 pounds.

Use the information from the above paragraphs to answer these questions. Write a mathematical sentence for each question first.

1. What is the weight of a gallon of milk? ($n = 1.03 \times 8.36$ The weight of a gallon of milk is 8.61 pounds.)
2. What will be the weight of the 10-gallon tank filled with water? ($n = (10 \times 8.36) + 18.7$, The weight of a 10-gallon tank filled with water is 102.3 pounds.)
3. What is the weight of 2 gallons of gasoline? ($n = 2 \times .67 \times 8.36$ The weight of 2 gallons of gasoline is 11.20 pounds.)
4. What is the weight of 1 quart of water? ($n = \frac{1}{4} \times 8.36$ One quart of water weighs 2.09 pounds.)
5. What is the weight of the 5-gallon can filled with turpentine? ($n = (5 \times .87 \times 8.36) + 10.4$ A 5-gallon can filled with turpentine will be 46.76 pounds.
6. 5 gallons of milk weighs how much more than 5 gallons of water? ($[(1.03 \times 8.36) \times 5] - [8.36 \times 5] = n$ Five gallons of milk weighs 1.25 pounds more than 5 gallons of water.)
7. Which weighs more, 7 gallons of gasoline or 5 gallons of turpentine? ($.67 \times 8.36 \times 7 = 39.20$ Seven gallons of gasoline weighs more
$.87 \times 8.36 \times 5 = 36.36$ than five gallons of turpentine.)
8. Which weighs more, 5 gallons of milk and 5 gallons of gasoline, or 10 gallons of turpentine?
($(1.03 \times 8.36 \times 5) + (.67 \times 8.36 \times 5) = 43.05 + 28 = 71.05$
$.87 \times 8.36 \times 10 = 72.7$
Ten gallons of turpentine weighs more.)

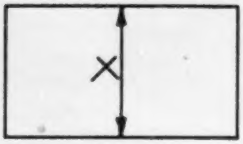
Exercise Set 16

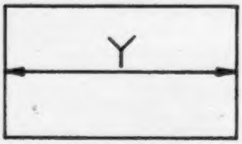
Which of the mathematical sentences in the box can be used to answer each question below? Solve the mathematical sentence and write an answer sentence.

a. $8 + n = 59.2$
b. $8 \times n = 59.2$
c. $n = 59.2 \times 8$
d. $n - 8 = 59.2$
e. $8 + n = 59.2$
f. $8 - n = 59.2$
g. $n + 8 = 59.2$

1. How many 8 inch pieces of pipe can be cut from a pipe whose length is 59.2 inches? (b. $8 \times n = 59.2$ Seven 8 inch pieces o pipe can be cut from a pipe whose length is 59.2 inches.)
2. A piece of pipe 59.2 inches long is how much longer than a piece 8 inches long? (e. $8 + n = 59.2$ A piece of pipe 59.2 inches long is 51.2 inches longer than a piece 8 inches long.)
3. What is the total length of eight pieces of pipe each 59.2 inches long? (c. $n = 59.2 \times 8$ The total length of eight pieces of pipe each 59.2 inch long is 473.6 inches.)
4. From what length of pipe is an 8 inch piece cut so the resulting piece is 59.2 inches long? (d. $n - 8 = 59.2$ An 8 inch piece of pipe is cut from a piece 67.2 inches long so the resulting piece is 59.2 inches long.)
5. What is the entire length of a piece of pipe if $\frac{1}{8}$ of its length is 59.2 inches? (g. $n \div 8 = 59.2$ The entire length of the pipe is 473.6 inches.)
6. It takes 2 minutes to make each cut thru a piece of pipe. How long does it take to cut the pipe into 2 pieces? (2 minutes. You need to make only one cut.)
7. Answer Exercise 6 for 3 pieces (4 minutes); for 4 pieces (6 minutes); for 5 pieces (8 minutes); for 10 pieces (18 minutes); for n pieces if n is a counting number. $[(n-1) \times 2]$

8. You can cut a rectangular region into parts by vertical cuts like x or horizontal cuts like y. It takes 2 minutes to make a vertical cut and 3 minutes to make a horizontal cut. How long would it take to cut the rectangle into four equal parts if





a. You could use only vertical cuts?
((4-1) × 2 = n It would take 6 minutes.)

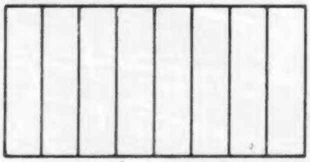
b. You could use only horizontal cuts?
((4-1) × 3 = n It would take 9 minutes.)

9. Use the information in Exercise 8 to answer these questions. You may use both horizontal and vertical cuts in answering.

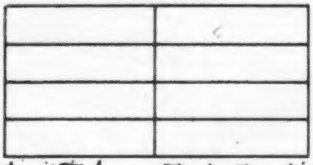
a. What is the shortest time needed to cut the rectangle into four equal parts? (By making one vertical cut and one horizontal cut it would take 3+2 or 5 minutes.)

b. What is the shortest time needed to cut the rectangle into 8 equal parts? (By making three vertical cuts and one horizontal cut, it would take (3×2)+3 or 9 minutes.)

c. Draw a picture to show how the cuts would be made in dividing the rectangle into 8 equal parts in 14 minutes; in 11 minutes.



(It will take 7 vertical cuts and 14 minutes to divide the rectangle into 8 equal parts.)



(It will take 1 vertical cut and 3 horizontal cuts and 11 minutes to divide the rectangle into 8 parts.)

Exercise Set 17

The expression $2 \times n + 1$ means, "Add 1 to the product of $2 \times n$. If $n = \frac{5}{3}$, then $2 \times n + 1$ is $(2 \times \frac{5}{3}) + 1 = \frac{10}{3} + 1 = \frac{13}{3}$."

Copy and fill in the blanks in these tables. In Exercise 1 you think, "If $n = 3$, then $2n + 1 = (2 \times 3) + 1 = 7$." Write 7 in the table below 3.

1. If $n =$	3	8	$2\frac{1}{2}$	$5\frac{1}{3}$	$\frac{11}{8}$	0	$\frac{7}{2}$
Then $2 \times n + 1 =$	7	17	6	$11\frac{2}{3}$	$\frac{15}{4}$ or $3\frac{3}{4}$	1	8

2. If $n =$	4	$1\frac{1}{2}$	$2\frac{1}{4}$	$5\frac{5}{6}$	$\frac{9}{7}$	$\frac{11}{3}$	8
Then $3n - 2 =$	10	$\frac{5}{2}$ or $2\frac{1}{2}$	$\frac{19}{4}$ or $4\frac{3}{4}$	$\frac{31}{2}$ or $15\frac{1}{2}$	$\frac{13}{7}$ or $1\frac{6}{7}$	9	22

3. If $n =$	2	6	3.2	4.6	.12	.78
Then $2.4n + 1.5 =$	6.3	15.9	9.18	12.54	1.788	3.372

4. If $n =$	3	.4	.8	100	.12	1.44
Then $(3.6 \div n) + .6 =$	1.8	9.6	5.1	0.636	30.6	3.1

5. If $n =$	5	12	.8	.94	1.8	3.9
Then $8.48 - (n \times .6) =$	5.48	1.28	8	7.916	7.40	6.14

6. If $n =$	.96	.144	.1728	.888	.0056
Then $12.72 - (n \div .08) =$	0.72	10.92	10.56	1.62	12.65

Exercise Set 18

The square on the right is called a multiplication magic square. The product of the numbers in each row and each column is the same: for example,
$2^4 \times 2^9 \times 2^2 = 2^{15}$ $2^9 \times 2^5 \times 2^1 = 2^{15}$

2^4	2^3	2^8
2^9	2^5	2^1
2^2	2^7	2^6

1. Copy the magic square on the right. Fill in all empty cells so it is a multiplicative magic square.

3^3	3^{10}	3^5
3^8	3^6	3^4
3^7	3^2	3^9

2. You are to make a multiplicative magic square from the one at the right. Do this by changing the position of two numbers.

	4^7	4^{14}	4^9	
4^{12}	4^{13}	4^{10}	4^8	
	4^{11}	4^6	4^{12}	4^{13}

3. Is this a multiplicative magic square? (yes) Study it before you start to multiply. (you may want to change it.)

16	8	256
512	32	2
4	128	64

2^4	2^3	2^8
2^9	2^5	2^1
2^2	2^7	2^6

4. Is this a multiplicative magic square? (yes)

.32	.16	5.12
10.24	.64	.04
.08	2.56	1.28

5. BRAINTWISTER. Use the number whose numerals are 5^3, 5^4, 5^5, 5^6, 5^7, 5^8, 5^9, 5^{10}, 5^{11} to make a multiplicative magic square. Hint: The product of the numbers in each row and column is 5^{21}.

(

5^6	5^5	5^{10}
5^{11}	5^7	5^3
5^4	5^9	5^8

Answers will vary.)

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